Semantics and Domain theory

Exercises 3

1. Do exercise 2.3.1 of Winskel at the end of Chapter 2, that is prove that the set of partial functions from X to Y, $X \rightharpoonup Y$, forms a domain, with the definitions of ordering and lub given on slide 10 (or the equivalent definitions given at the lecture).

2. Which of the following is a domain? (In each case, choose a proper definition of 'lub'; prove your answer.)

   (a) $(\mathcal{P}(X), \subseteq)$, where $\mathcal{P}(X)$ is the powerset of X and $\subseteq$ is the usual subset ordering.

   (b) $([0,1], \leq)$, where $[0,1]$ is the unit interval and $\leq$ is the usual ordering on the real numbers.

   (c) $([0,1] \cap \mathbb{Q}, \leq)$, where $\cap$ is the intersection and $\mathbb{Q}$ is the set of rational numbers.

   (d) $(\Sigma^*, \sqsubseteq)$, where Σ^* is the set of words over the alphabet $\Sigma := \{a, b\}$ and $\sqsubseteq$ is the prefix ordering, defined by $w \sqsubseteq wv$ for all $w, v \in \Sigma^*$.

   (e) $(\Sigma^* \cup \Sigma^\omega, \sqsubseteq)$, where Σ^ω is the set of infinite words over the alphabet $\Sigma := \{a, b\}$ and $\sqsubseteq$ is the prefix ordering, defined by $w \sqsubseteq wv$ for all $w \in \Sigma^*$, $v \in \Sigma^* \cup \Sigma^\omega$ and $v \sqsubseteq v$ for all $v \in \Sigma^* \cup \Sigma^\omega$.

3. Do exercise 2.3.2 of Winskel at the end of Chapter 2, that is prove that the function $f_{b,c}$ in the definition of the denotational semantics of **while** B **do** C is continuous.

4. Let $(d_i)_{i \geq 0}$ and $(e_i)_{i \geq 0}$ be chains in a domain $(D, \sqsubseteq)$. Suppose that $(d_i)_{i \geq 0}$ is *majorized by* $(e_i)_{i \geq 0}$, that is: $\forall\, i\, \exists\, j (d_i \sqsubseteq e_j)$.
   Prove that $\sqcup_{i \geq 0} d_i \sqsubseteq \sqcup_{i \geq 0} e_i$.

5. Suppose that in the domain $(D, \sqsubseteq)$, all chains are *eventually constant*, that is: for all chains $(d_i)_{i \geq 0}$ there exists an n such that $d_n = d_{n+1} = d_{n+2} = \ldots$
   Show that every monotone $f : D \to D$ is continuous.