EXAM 2014 (Semantics & Domain Theory) ①

1.a $[\![S]\!] = \mathrm{Fix}(F)$ with $F: (\mathrm{State} \to \mathrm{State}) \to (\mathrm{State} \to \mathrm{State})$ given by

$$F(w)(s) = \mathrm{if}\left([\![x>0]\!](s),\ w(s[y \mapsto s(y)+s(x),\ x \mapsto s(x)-1]),\ s\right)$$

$$= \begin{cases} s & \text{if } s(x) \leq 0 \\ w(s[y \mapsto s(y)+s(x),\ x \mapsto s(x)-1]) & \text{if } s(x) > 0 \end{cases}$$

For $s \in \mathrm{State}$:

$$F(w_n)(s) = \begin{cases} s & \text{if } s(x) \leq 0 \\ w_n(s[y \mapsto s(y)+s(x),\ x \mapsto s(x)-1]) & \text{if } s(x) > 0 \end{cases}$$

We consider the case $s(x) > 0$: then

$$= \begin{cases} s[y \mapsto s(y)+s(x),\ x \mapsto s(x)-1] & \text{if } s(x)-1 \leq 0 \\ s\left[x \mapsto 0,\ y \mapsto s(y)+s(x) + \frac{(s(x)-1)\,s(x)}{2}\right] & \text{if } 0 < s(x)-1 < n \\ \perp & \text{if } n \leq s(x)-1 \end{cases}$$

$$= \begin{cases} s[y \mapsto s(y)+s(x)\ \ x \mapsto 0] & \text{if } s(x) = 1 \\ s\left[x \mapsto 0,\ y \mapsto s(y) + \frac{2s(x)+s(x)^2-s(x)}{2}\right] & \text{if } 1 < s(x) < n+1 \\ \perp & \text{if } n+1 \leq s(x) \end{cases}$$

$$= \begin{cases} s\left[x \mapsto 0,\ y \mapsto s(y) + \frac{s(x)\cdot(s(x)+1)}{2}\right] & \text{if } s(x) = 1 \\ s\left[x \mapsto 0,\ y \mapsto s(y) + \frac{s(x)\,(s(x)+1)}{2}\right] & \text{if } 1 < s(x) < n+1 \\ \perp & \text{if } n+1 \leq s(x) \end{cases}$$

$$= w_{n+1}(s)$$

Also if $s(x) \leq 0$ we have $w_{n+1}(s) = F(w_n)(s)$, so $F(w_n) = w_{n+1}$.

1b $W_\infty = \bigsqcup_{n \in \mathbb{N}} W_n$ is given by

$$W_\infty(s) = \begin{cases} s & \text{if } s(x) \leq 0 \\ s\left[x \mapsto 0,\ y \mapsto s(y) + \frac{s(x)\cdot(s(x)+1)}{2}\right] & \text{if } s(x) > 0 \end{cases}$$

$$F(W_\infty)(s) = \begin{cases} s & \text{if } s(x) \leq 0 \\ W_\infty\left(s[y \mapsto s(y) + s(x),\ x \mapsto s(x)-1]\right) & \text{if } s(x) > 0 \end{cases}$$

If $s(x) \leq 0$: $W_\infty(s) = F(W_\infty)(s)$

If $s(x) > 0$:

Case $s(x) = 1$:
$$\begin{aligned} F(W_\infty)(s) &= W_\infty\left(s[y \mapsto s(y)+1,\ x \mapsto 0]\right) \\ &= s[y \mapsto s(y)+1,\ x \mapsto 0] \\ &= s\left[x \mapsto 0,\ y \mapsto s(y) + \tfrac{1\cdot(1+1)}{2}\right] \\ &= W_\infty(s) \end{aligned}$$

Case $s(x) > 1$:
$$\begin{aligned} F(W_\infty)(s) &= W_\infty\left(s[y \mapsto s(y) + s(x),\ x \mapsto s(x)-1]\right) \\ &= s\left[x \mapsto 0,\ y \mapsto s(y) + s(x) + \tfrac{(s(x)-1)\cdot s(x)}{2}\right] \\ &= s\left[x \mapsto 0,\ y \mapsto s(y) + \tfrac{s(x)\cdot(s(x)+1)}{2}\right] \\ &= W_\infty(s) \end{aligned}$$

So $F(W_\infty)(s) = W_\infty(s)$.

4a $V \Rightarrow 0$ Abbreviate: $M = \text{if zero } x \text{ then } x \text{ else } P_1 \,(\text{pred } x)$

$F = \text{fn } f.\ \text{fn } x.\ \text{if zero } x \text{ then } x \text{ else } f\,(\text{pred } x)$

$$\dfrac{\dfrac{\dfrac{F \Downarrow F \qquad \text{fn } x.\ \text{if zero } x \text{ then } x \text{ else } P_1(\text{pred } x)}{F\,P_1 \Downarrow \text{fn } x.M}}{P_1 \Downarrow \text{fn } x.\ M} \qquad \dfrac{\text{zero}(\text{succ } 0) \Downarrow \text{false} \qquad \dfrac{\dfrac{\vdots}{P_1 \Downarrow \text{fn } x.M} \qquad \overset{(*)}{M[{}^{\text{pred}(\text{succ } 0)}/_{x}] \Downarrow 0}}{P_1(\text{pred}(\text{succ } 0)) \Downarrow 0}}{M[{}^{\text{succ } 0}/_{x}] \Downarrow 0}}{P_1\,(\text{succ } 0) \Downarrow 0}$$

$$(*):\quad \dfrac{\dfrac{\dfrac{\vdots}{\text{pred}(\text{succ } 0) \Downarrow 0}}{\text{zero}(\text{pred}(\text{succ } 0)) \Downarrow \text{true}} \qquad \dfrac{\dfrac{}{\text{succ } 0 \Downarrow \text{succ } 0}}{\text{pred}(\text{succ } 0) \Downarrow 0}}{M[{}^{\text{pred}(\text{succ } 0)}/_{x}] \Downarrow 0}$$

4b ~~$[\![P_1]\!] = \text{fix } \lambda f \in \mathbb{N}_\perp \to \mathbb{N}_\perp .\ \lambda x \in \mathbb{N}_\perp .\ \text{if}$~~

$[\![P_1]\!] = \text{fix } F$ with

$$F = \lambda f \in \mathbb{N}_\perp \to \mathbb{N}_\perp .\ \lambda x \in \mathbb{N}_\perp .\ \begin{cases} \perp & \text{if } x = \perp \\ 0 & \text{if } x = 0 \\ f(x-1) & \text{if } x \in \mathbb{N},\ x > 0 \end{cases}$$

so $[\![P_1]\!] = \text{fix } F = \bigsqcup_{n \in \mathbb{N}} F^n(\perp)$ ~~$= \begin{cases} \perp & \text{if } x = \perp \\ 0 & \text{if } x \in \mathbb{N} \end{cases}$~~

$$= \lambda x \in \mathbb{N}_\perp .\ \begin{cases} \perp & \text{if } x = \perp \\ 0 & \text{if } x \in \mathbb{N} \end{cases}$$

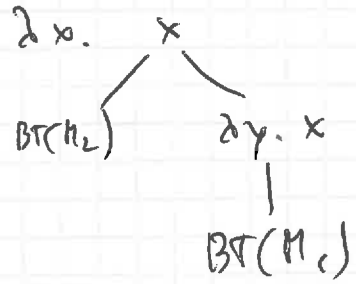
4c $[\![P_2]\!] = \lambda x \in \mathbb{N}_\perp .\ 0$

$[\![P_3]\!] = \lambda x \in \mathbb{N}_\perp .\ \begin{cases} \perp & \text{if } x = \perp \\ 0 & \text{if } x \in \mathbb{N} \end{cases}$

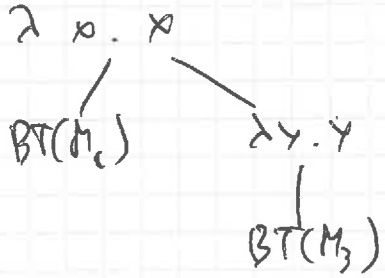
so $[\![P_1]\!] = [\![P_3]\!]$, $[\![P_2]\!] \neq [\![P_1]\!], [\![P_3]\!]$

5

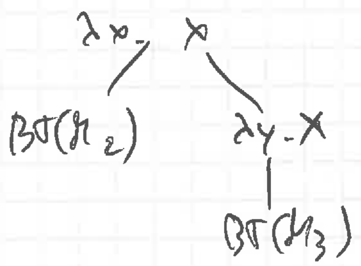
$BT(M_1)$:



$BT(M_2)$:



$BT(M_3)$:



M_2 has a different BT from M_1, M_3 so $D_A \not\models M_2 = M_1$

$D_A \not\models M_2 = M_3$

M_1 and M_3 have the same Böhm tree

so $D_A \models M_1 = M_3$