Semantics and Domain theory

Exercises 1

At the lecture, we gave a denotational semantics for the language L given by the grammar

$$\begin{array}{rcl} b : \mathsf{bit} & ::= & \mathbf{0} \mid \mathbf{1} \\ n : \mathsf{bin} & ::= & b \mid n\, b \end{array}$$

NB **0** and **1** are symbols, not the numbers.

The semantics is given by the model $\mathbf{N}$, the natural numbers, and the interpretation

$$\begin{array}{rcl} [\![\mathbf{0}]\!] & := & 0 \\ [\![\mathbf{1}]\!] & := & 1 \\ [\![n\, b]\!] & := & 2 * [\![n]\!] + [\![b]\!] \end{array}$$

In the lecture, we have recursively defined the operation $P(n)$, which prefixes a binary numeral n with a leading **0** as follows.

$$\begin{array}{rcl} P(\mathbf{0}) & := & \mathbf{0}\,\mathbf{0} \\ P(\mathbf{1}) & := & \mathbf{0}\,\mathbf{1} \\ P(n\, b) & := & P(n)\, b \end{array}$$

We have given an operational semantics $\overset{P}{\Longrightarrow}$ via the rules

$$\frac{}{\mathbf{0} \overset{P}{\Longrightarrow} \mathbf{0}\,\mathbf{0}} \qquad \frac{}{\mathbf{1} \overset{P}{\Longrightarrow} \mathbf{0}\,\mathbf{1}} \qquad \frac{n \overset{P}{\Longrightarrow} m}{n\, b \overset{P}{\Longrightarrow} m\, b}$$

Exercises:

1. Define the operation $S(n)$, which computes the binary numeral which is the successor of n.
2. (a) Give an operational semantics for $S(n)$, in the form of a relation $n \overset{S}{\Longrightarrow} m$ such that $S(n) = m$ iff $n \overset{S}{\Longrightarrow} m$
   (b) Prove that $S(n) = m$ iff $n \overset{S}{\Longrightarrow} m$
3. Prove $[\![S(n)]\!] = [\![n]\!] + 1$ for all n.
4. (a) Compute the denotational semantics of $S_1 :\equiv x := x + 1;\ y := x + x$
   (b) Compute the denotational semantics of $S_2 :\equiv$ if $x > 0$ then $x := 1$ else $x := -1$

   NB Your answer should be a "state transformers", i.e. an element of $\mathsf{State} \rightharpoonup \mathsf{State}$, the set of partial functions from State to State. For us a state is a function from variables to integers, $r : \mathsf{V} \to \mathbf{Z}$.