Semantics and Domain theory

Exercises 6

1. (This is basically Exercise 4.4.1. of DENS)
   (a) Show that if S and T are chain-closed, then $S \cup T$ is chain-closed.
   (b) Show that if S_i is chain-closed for every $i \in I$, then $\bigcap_{i \in I} S_i$ is chain-closed.

2. (Exercise 4.4.2.) Give an example of a subset S of $D \times D$ that is not chain-closed, but which satisfies:

$$\begin{array}{ll} (a) & \forall d \in D,\ \{d' | (d, d') \in S\} \text{ is chain-closed} \\ (b) & \forall d' \in D,\ \{d | (d, d') \in S\} \text{ is chain-closed.} \end{array}$$

   [Hint: consider $D = D = \Omega$, the cpo in Figure 1.] (Compare this with the property of continuous functions given on Slide 15.)

3. The collection of chain-closed sets is not closed under arbitrary union. (It is not the case, in general that $\forall i \in I(S_i$ is chain closed) implies $\bigcup_{i \in I} S_i$ is chain-closed.)
   (a) Conclude this from the previous exercise.
   (b) Conclude this by directly consructing a counterexample in Ω.

4. Let $P : D \to B_\perp$ and $g : D \to D$ be continuous. Define $f : D \times D \to D \times D$ by

$$f(d_1, d_2) = \text{if}(P(d_1), (g(d_1), g(d_2)), (g(d_2), g(d_1))).$$

   Show that for $\text{fix}(f) = (u_1, u_2)$, we have $u_1 = u_2$. (Use Scott induction.)

5. Prove that for $f : D \to E$ monotone,

$$f^{-1} \text{ preserves chain-closed sets} \Rightarrow f \text{ is continuous},$$

   where f^{-1} *preserves chain-closed sets* means that, for all $S \subseteq E$, if S is chain-closed, then $f^{-1}(S)$ is a chain-closed subset of D.