Semantics and Domain theory

Exercises 10

1. (Exercise 8.4.1) Suppose that a monotonic function $p : (\mathbb{B}_\perp \times \mathbb{B}_\perp) \to \mathbb{B}_\perp$ satisfies

   - $p(\mathsf{tt}, \perp) = \mathsf{tt}$,
   - $p(\perp, \mathsf{tt}) = \mathsf{tt}$,
   - $p(\mathsf{ff}, \mathsf{ff}) = \mathsf{ff}$.

   Show that p coincides with the parallel-or function on Slide 45 in the sense that $p(d_1, d_2) = \mathrm{por}(d_1)(d_2)$, for all $d_1, d_2 \in \mathbb{B}_\perp$.

2. (Exercise 7.4.2.) For any PCF type τ and closed terms M_1, M_2 of type τ, we have

   $$(\forall V : \tau, (M_1 \Downarrow_\tau V \Leftrightarrow M_2 \Downarrow_\tau V)) \Rightarrow M_1 \cong_{\mathrm{ctx}} M_2 : \tau. \qquad (**)$$

   Use (**) to show that β-conversion is valid up to contextual equivalence in PCF, in the sense that for all closed terms $\mathbf{fn}\, x : \tau_1.\, P : \tau_1 \to \tau_2$ and $Q : \tau_1$,

   $$(\mathbf{fn}\, x : \tau_1.\, P)\, Q \cong_{\mathrm{ctx}} P[Q/x] : \tau_2.$$

3. (Exercise 7.4.3.) We show that the converse of (**) is not valid at all types

   (a) Consider the terms $M_1 := \mathbf{fix}(\mathbf{fn}\, f : \mathbf{nat} \to \mathbf{nat}.f)$ and $M_2 := \mathbf{fn}\, x : \mathbf{nat}.\mathbf{fix}(\mathbf{fn}\, x : \mathbf{nat}.x)$ of type $\mathbf{nat} \to \mathbf{nat}$ and use the extensionality property of $\leq_{\mathrm{ctx}}$ at function types (Slide 44) to show that $M_1 \cong_{\mathrm{ctx}} M_2$.

   (b) Show that the left hand side of (**) does not hold for these terms M_1 and M_2.