Semantics and Domain theory
Exercises 12

1. Prove the correctness of Definition 3.2.5. To prove this, you have to show that the function
$$\boldsymbol{\lambda} d.[\![P]\!]_{\rho(x:=d)}$$
is continuous for every P and ρ. (You may assume that F and G are continuous and all the other results about continuity from the notes.)

2. At the lecture, we have seen the interpretations in D_A of $\mathbf{I}$ ($= \lambda x.x$), $\mathbf{K}$ ($= \lambda x.\lambda y.x$) and $\mathbf{II}$.
   (a) Compute the interpretation of $\lambda x.x\,x$.
   (b) Show that $[\![\mathbf{KI}]\!] = \{(\beta, (\gamma, c)) \mid c \in \gamma\}$ (without doing a β-reduction first).

3. Let Y be an element of D_A and let ρ be a valuation with $\rho(y) = Y$.
   (a) Compute in D_A the interpretation of $\lambda x.y\,x$ by expressing $[\![\lambda x.y\,x]\!]_\rho$ in terms of Y.
   (b) Conclude that the η-rule does not hold in D_A. (The η-rule says that $\lambda x.M\,x = M$ if $x \notin \mathrm{FV}(M)$.)

4. Use the result of the following exercise ($[\![\Omega]\!] = \emptyset$) to
   (a) compute the interpretation of $\lambda y.\Omega$ in D_A,
   (b) compute the interpretation of $\lambda y.y\,\Omega$ in D_A.

5. [Challenging] Show that the interpretation of Ω ($= (\lambda x.x\,x)(\lambda x.x\,x)$) in D_A is $\emptyset$.
   (Hint: From a $c \in [\![\Omega]\!]$ you can construct an infinite sequence $(\alpha_i)_{i\in\mathbb{N}}$ with $(\alpha_{i+1}, c) \in \alpha_i$ for all i, which is impossible in D_A.)

6. Prove that, for M a closed λ-term, if M has a head-normal-form, then there is a sequence of terms $P_1, \ldots, P_n$ such that $M\,P_1 \ldots P_n =_\beta \mathbf{I}$.
   (For closed terms, the reverse implication also holds, so this criterion is equivalent to *having a hnf*. This is where the terminology *solvable* comes from.)