Semantics and Domain theory

Exercises 5

1. Prove Proposition 3.3.1, that is prove that, given the partial function $f : X \rightharpoonup Y$, the function $f_\perp : X_\perp \to Y_\perp$ is continuous.

2. Prove Proposition 3.3.2, that is, prove that for each domain D the function $\mathrm{if} : B_\perp \times (D \times D) \to D$ defined by $\mathrm{if}(\mathrm{tt}, (d, e)) = d$, $\mathrm{if}(\mathrm{ff}, (d, e)) = e$ and $\mathrm{if}(\perp, (d, e)) = \perp$ is continuous.

3. (Exercise 3.4.2 of Winskell): Let X and Y be sets and $X_\perp$ and $Y_\perp$ be the corresponding flat domains. Show that a function $f : X_\perp \to Y_\perp$ is continuous if and only if one of (a) or (b) holds:

   (a) f is strict, i.e. $f(\perp) = \perp$.

   (b) f is constant, i.e. $\forall\, x \in X(f(x) = f(\perp))$.

4. Show that the following two definitions of the ordering between functions $f, g : D \to E$ (see Slide 17) are equivalent.

   (a) $f \sqsubseteq g := \forall\, d \in D(f(d) \sqsubseteq_E g(d))$.

   (b) $f \sqsubseteq' g := \forall\, d_1, d_2 \in D(d_1 \sqsubseteq_D d_2 \Rightarrow f(d_1) \sqsubseteq_E g(d_2))$.

5. Prove that for D a domain and $F : (D \to D) \to (D \to D)$ and $g : D \to D$ continuous,

$$\mathrm{ev}(\mathrm{fix}(F), \mathrm{fix}(g)) = \sqcup_{k \geq 0} F^k(\perp')(g^k(\perp)),$$

   where $\perp$ is in D and $\perp'$ is in $D \to D$ and ev is the evaluation function of Proposition 3.2.1.

6. We define two variants of a functional $F : [\mathbb{N}_\perp \to \mathbb{N}_\perp] \to \mathbb{B}_\perp$ that, given a continuous function $f : \mathbb{N}_\perp \to \mathbb{N}_\perp$, checks if f has a fixed point in $\mathbb{N}$. (That is: an $n \in \mathbb{N}$ for which $f(n) = n$).

$$F_1(f) \quad := \quad \begin{cases} \mathbf{tt} & \text{if } f \text{ is total and } \exists\, n \in \mathbb{N}(f(n) = n) \\ \mathbf{ff} & \text{if } f \text{ is total and } \forall\, n \in \mathbb{N}(f(n) \neq n) \\ \perp & \text{if } f \text{ is not total} \end{cases}$$

$$F_2(f) \quad := \quad \begin{cases} \mathbf{tt} & \text{if } \exists\, n \in \mathbb{N}(f(n) = n \wedge \forall\, m < n(f(m) \neq m, \perp)) \\ \perp & \text{otherwise} \end{cases}$$

   NB. "f is total" means that $\forall\, n \in \mathbb{N}(f(n) \neq \perp)$.

   You may assume that both F_1 and F_2 are monotone: this has been checked in the lecture. (If it hasn't, please verify this for yourself.)

   (a) Prove that one of the F_i does not preserve lubs (and thus is not continuous).

   (b) Prove that one of the F_i preserves lubs (and thus is continuous).