Semantics and Domain theory
Exercises 9

1. Prove Theorem 6.4.1 for the inductive cases $\Downarrow_{\text{pred}}$ and $\Downarrow_{\text{if1}}$.
   Remember that Theorem 6.4.1 states that for all closed expressions M and V and type τ, if $M \Downarrow_\tau V$, then $[\![M]\!] = [\![V]\!]$. It is proved by induction on the derivation of $M \Downarrow_\tau V$.

2. Prove the following properties (by induction on τ). Here, M, M_1, M_2 range over closed terms, d_1, d_2 are domain elements.

   (a) If $d_2 \sqsubseteq d_1$ and $d_1 \lhd_\tau M_1$, then $d_2 \lhd_\tau M_1$.

   (b) If $d_1 \lhd_\tau M_1$ and $\forall V(M_1 \Downarrow_\tau V \Rightarrow M_2 \Downarrow_\tau V)$, then

   $$d_1 \lhd_\tau M_2$$

   These properties constitute Lemma 7.2.1 (iii).

3. Remember that $\lhd_\tau$ denotes the approximation relation (slide 39).
   Show that, if $d \lhd_{\mathbf{nat}} M$, $e \lhd_{\mathbf{nat}} N$ and $b \lhd_{\mathbf{bool}} P$, then

   $$\text{if}(b, d, e) \lhd_{\mathbf{nat}} \mathbf{if}\ P\ \mathbf{then}\ M\ \mathbf{else}\ N$$

   (This is the "if" inductive case in the proof of the Fundamental Property, Slide 40)

4. Prove that $\mathbf{fn}\, x : \mathbf{nat}.\, \mathbf{succ}(\mathbf{pred}\, x) \leq_{\text{ctx}} \mathbf{fn}\, x : \mathbf{nat}.x$ by using the Extensionality properties on Slide 44.