TEST EXAM S & DT

1a $F(w)(s) = if\left([\![x>0]\!](s),\ w\left([\![x:=x-1;\ y:=y+z]\!]\right)(s),\ skip(s) \right)$

$$= \begin{cases} s & \text{if } s(x) \le 0 \\ w(s[x\mapsto s(x)-1,\ y\mapsto s(y)+s(z)]) & \text{if } s(x)>0 \end{cases}$$

With w_n given as in the exercise:

$$F(w_n)(s) = \begin{cases} s & \text{if } s(x)\le 0 \\ w_n(s[x\mapsto s(x)-1,\ y\mapsto s(y)+s(z)]) & \text{if } s(x)>0 \end{cases}$$

To prove $F(w_n) = w_{n+1}$ we prove $F(w_n)(s) = w_{n+1}(s)$ $(\forall s)$

let s be a state. If $s(x)\le 0$, then $F(w_n)(s) = w_{n+1}(s) = s$ ✓

If $s(x)>0$:

$$F(w_n)(s) = w_n(s[x\mapsto s(x)-1,\ y\mapsto s(y)+s(z)]) =$$

$$= \begin{cases} s[x\mapsto s(x)-1,\ y\mapsto s(y)+s(z)] & \text{if } s(x)=1 \\ s[x\mapsto 0,\ y\mapsto s(y)+s(z)+(s(x)-1)*s(z)] & \text{if } 1<s(x)<n+1 \\ \perp & \text{if } s(x)\ge n+1 \end{cases}$$

$$= \begin{cases} s[x\mapsto 0,\ y\mapsto s(y)+s(x)*s(z)] & \text{if } s(x)=1 \\ s[x\mapsto 0,\ y\mapsto s(y)+s(x)*s(z)] & \text{if } 1<s(x)<n+1 \\ \perp & \text{if } s(x)\ge n+1 \end{cases}$$

$$= w_{n+1}(s)$$

$\boxtimes$

1b $\quad w_\infty = \bigsqcup_{i \in \mathbb{N}} w_i$ ②

So $w_\infty(s) = \bigsqcup_{i \in \mathbb{N}} w_i(s)$. Let $n \in \mathbb{N}$.

- If $s(x) \leq 0$: $w_\infty(s) = \bigsqcup_{i \in \mathbb{N}} w_i(s) = s$
- If $s(x) \geq n$: $w_n(s) = \bot$
- If $s(x) < n$: $w_n(s) = s[x \mapsto 0, y \mapsto s(y) + s(x) * s(z)]$

So, if $s(x) = p > 0$, then $w_0(s) = \dots = w_p(s) = \bot$

and $w_{p+1}(s) = w_{p+2}(s) = \dots = s[x \mapsto 0, y \mapsto s(y) + s(x) * s(z)]$

So $\bigsqcup_{i \in \mathbb{N}} w_i(s) = s[x \mapsto 0, y \mapsto s(y) + s(x) * s(z)]$

So $w_\infty(s) = \begin{cases} s & \text{if } s(x) \leq 0 \\ s[x \mapsto 0, y \mapsto s(y) + s(x) * s(z)] & \text{if } s(x) > 0 \end{cases}$

$$F(w_\infty)(s) = \begin{cases} s & \text{if } s(x) \leq 0 \\ w_\infty(s[x \mapsto s(x) - 1, y \mapsto s(y) + s(z)]) & \text{if } s(x) > 0 \end{cases}$$

$$= \begin{cases} s & \text{if } s(x) \leq 0 \\ s[x \mapsto 0, y \mapsto s(y) + s(z)] & \text{if } s(x) = 1 \\ s[x \mapsto 0, y \mapsto s(y) + s(z) + (s(x) - 1) * s(z))] & \text{if } s(x) > 1 \end{cases}$$

$$= \begin{cases} s & \text{if } s(x) \leq 0 \\ s[x \mapsto 0, y \mapsto s(y) + s(z) * s(x)] & \text{if } s(x) = 1 \\ s[x \mapsto 0, y \mapsto s(y) + s(z) * s(x)] & \text{if } s(x) > 1 \end{cases}$$

$$= w_\infty$$ $\boxtimes$

2 For $f, g \in (D, \sqsubseteq) \xrightarrow{mon} (D, \sqsubseteq)$

$f \sqsubseteq g \quad := \quad \forall x \in D \, (f(x) \sqsubseteq g(x))$

a) This is a p.o : (refl) $f \sqsubseteq f$

(antisym) $f \sqsubseteq g \wedge g \sqsubseteq f \rightarrow f = g$

(trans) $f \sqsubseteq g \wedge g \sqsubseteq h \rightarrow f \sqsubseteq h$

b) Every chain has a lub. Let $(f_i)_{i \in \mathbb{N}}$ be a chain

$\bigsqcup_{i \in \mathbb{N}} f_i$ is defined by

$$\left(\bigsqcup_{i \in \mathbb{N}} f_i\right)(x) := \bigsqcup_{i \in \mathbb{N}} (f_i(x))$$

- The lub in the right hand side exists, because

$f_0(x) \sqsubseteq f_1(x) \sqsubseteq f_2(x) \sqsubseteq f_3(x)$

due to the fact that $f_0 \sqsubseteq f_1 \sqsubseteq f_2$ --

~~The RHS is an upper bound~~

- $\bigsqcup_{i \in \mathbb{N}} f_i$ is an upper bound of $(f_i)_{i \in \mathbb{N}}$ because

for all $x \in D$: $f_i(x) \sqsubseteq \bigsqcup_{i \in \mathbb{N}} f_i(x)$

- $\bigsqcup_{i \in \mathbb{N}} f_i$ is the least upperbound of $(f_i)_{i \in \mathbb{N}}$ because,

for g an upperbound of $(f_i)_{i \in \mathbb{N}}$,

~~the thank~~ let $x \in D$, then

$\forall i \quad f_i(x) \sqsubseteq g(x)$

so $\bigsqcup_{i \in \mathbb{N}} f_i(x) \sqsubseteq g(x)$ (because $\bigsqcup_{i \in \mathbb{N}} f_i(x)$ is a lub)

so $\left(\bigsqcup_{i \in \mathbb{N}} f_i\right)(x) \sqsubseteq g(x)$

this for all $x \in D$. So $\bigsqcup_{i \in \mathbb{N}} f_i \sqsubseteq g$. $\boxtimes$

3a To prove $\forall i\, (\underbrace{k_i \sqsubseteq k_{i+1}}_{\forall x \in \mathbb{N}_\perp (k_i(x) \sqsubseteq k_{i+1}(x))})$

Let $i \in \mathbb{N}$, $x \in \mathbb{N}_\perp$

- if $x = \perp$, then $k_i(x) = \perp$ so ✓
- if $x \notin D_i$, then $k_i(x) = \perp$, so ✓
- if $x \in D_i$, then $k_i(x) = p$

  $D_i \subseteq D_{i+1}$, so $x \in D_{i+1}$, so $k_{i+1}(x) = p$ ✓

3b i To prove: if $f \sqsubseteq g$, then $H(f) \sqsubseteq H(g)$

Suppose $f \sqsubseteq g$

Cases $H(f) = 0$. (Then $\forall x \in \mathbb{N}_\perp\ f(x) = \perp$)

then $H(f) \sqsubseteq H(g)$ ✓

Case $H(f) = \frac{1}{n+1}$. Then $f(n) \neq \perp$ and $\forall i < n\ (f(i) = \perp)$

From $f \sqsubseteq g$: $g(n) = f(n) \neq \perp$

This means that for the smallest m for which $g(m) \neq \perp$

we know $m \leq n$

So $H(g) = \frac{1}{m+1} \geq \frac{1}{n+1} = H(f)$ ✓

3b ii The k_i functions are given by D_i, as in 3a.

$\bigsqcup_{i \in \mathbb{N}} k_i : \mathbb{N}_\perp \to \mathbb{N}_\perp$ is the function K with

$$\begin{cases} K(x) = p & \text{if } x \in \bigcup_{i \in \mathbb{N}} D_i \\ K(x) = \perp & \text{if } x = \perp \text{ or } x \notin \bigcup_{i \in \mathbb{N}} D_i \end{cases}$$

So $H(\bigsqcup_{i \in \mathbb{N}} k_i) = H(K) = \begin{cases} 0 & \text{if } \forall i\ D_i = \emptyset \\ \frac{1}{m+1} & \text{if } m \text{ is the smallest elt in } \bigcup_{i \in \mathbb{N}} D_i \end{cases}$

3.b.iii We know that H is monotone (3.b. i)

Let $(f_i)_{i\in\mathbb{N}}$ be a chain in $\mathbb{N}_\perp \xrightarrow{mon} \mathbb{N}_\perp$

We prove that $H(\bigsqcup_{i\in\mathbb{N}} f_i) = \bigsqcup_{i\in\mathbb{N}} H(f_i)$

We introduce notation:

$$D_i := \{x \in \mathbb{N} \mid f_i(x) \neq \perp\}$$

(So D_i is more or less the domain of definedness of f_i)

Then $D_0 \subseteq D_1 \subseteq D_2 \ldots$ and

$\bigsqcup f_i$ is the function F that has the property

$$\begin{cases} F(x) \neq \perp & \text{if } x \in \bigcup_{i\in\mathbb{N}} D_i \\ F(x) = \perp & \text{if } x = \perp \text{ or } x \notin \bigcup_{i\in\mathbb{N}} D_i \end{cases}$$

So $H(\bigsqcup f_i) = \begin{cases} 0 & \text{if } \forall i\ (D_i = \emptyset) \\ \frac{1}{m+1} & \text{if } m \text{ is the smallest elt in } \bigcup_{i\in\mathbb{N}} D_i \end{cases}$

On the other hand:

$$H(f_i) = \begin{cases} 0 & \text{if } D_i = \emptyset \\ \frac{1}{n+1} & \text{if } n \text{ is the smallest elt in } D_i \end{cases}$$

So $\bigsqcup H(f_i) = \begin{cases} 0 & \text{if } \forall i\ (D_i = \emptyset) \\ \frac{1}{m+1} & \text{if } m \text{ is the smallest elt in } \bigcup_{i\in\mathbb{N}} D_i \end{cases}$

$\boxtimes$

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4a <u>monotone</u>: Suppose $(x,y) \sqsubseteq (x',y')$

we only have to consider the case where $f_i(x,y) \neq \bot$

Then $x \in \mathbb{N}$, $y \in \mathbb{N}$, so ~~also~~ $x' = x$ and $y = y'$

So $f_i(x,y) = f_i(x',y')$ $\boxtimes$

<u>continuous</u> f_i is continuous if $f_i^{1,x} : \mathbb{N}_\bot \to \mathbb{N}_\bot$ and $f_i^{2,y} : \mathbb{N}_\bot \to \mathbb{N}_\bot$ are continuous ~~with~~ for every $x, y \in \mathbb{N}_\bot$ where

$$f_i^{1,x}(y) := f_i(x,y)$$
$$f_i^{2,y}(x) := f_i(x,y)$$

f_i is monotone, so $f_i^{1,x}$ and $f_i^{2,y}$ are monotone ($\forall x, y$)

~~So~~ Now $f_i^{1,x}$ and $f_i^{2,y}$ are also continuous, because all chains in $\mathbb{N}_\bot$ are finite. So $\boxtimes$

(Can also be shown directly because all chains in $\mathbb{N}_\bot \times \mathbb{N}_\bot$ are finite! [<u>lemma</u> If all chains in D are finite, then $f: D \to E$ mon $\Rightarrow$ cont]

<u>4b</u> $\forall i \; f_i \sqsubseteq f_{i+1}$ by showing for given i: $\forall (x,y) \; f_i(x,y) \sqsubseteq f_{i+1}(x,y)$

<u>Proof</u> We only have to consider the case where $f_i(x,y) \neq \bot$

Then if $x = n \in \mathbb{N}$, $y = m \in \mathbb{N}$ with $f_i(n,m) = k$, the smallest $k \leq i$ with $k * n \geq m$

For this k we also have $k \leq i+1$,

so $f_{i+1}(n,m) = k$ as well, so $f_i(n,m) \sqsubseteq f_{i+1}(n,m)$ $\boxtimes$

<u>4c</u> let $F := \bigsqcup_{i \in \mathbb{N}} f_i$

Then $F(x,y) = \bot$ if $x = \bot$ or $y = \bot$

For $n, m \in \mathbb{N}$ $\underbrace{f_i(n,m) = \bot}_{\text{if } \forall k \leq i \, (k*n < m)}$ or $\underbrace{f_i(n,m) = k, \text{ the smallest } k \leq i \text{ such that } k*n \geq m}_{\text{if } \exists k \leq i \, (k*m \geq m)}$

So, if $\exists k . (k*n \geq m)$, then $f_i(n,m) = k$ for some i, the smallest k such that $k*n \geq m$;

We conclude:

$$F(x,y) = \begin{cases} \bot & \text{if } x = \bot \text{ or } y = \bot \text{ or } (x = 0 \text{ and } y > 0) \\ 0 & \text{if } y = 0 \\ \lceil \frac{y}{x} \rceil & \text{otherwise } (x > 0 \text{ and } y > 0) \end{cases}$$

5a $\preceq$ is monotone. Suppose $(x,y) \sqsubseteq (x',y')$

We only have to check the case when $\preceq(x,y) \neq \perp$.

Then $x,y \in \mathbb{N}$, so $x'=x$, $y'=y$ and $\preceq(x,y) = \preceq(x',y')$ $\boxtimes$

$\preceq$ is continuous because it is monotone on a domain where all chains are finite $\boxtimes$

5b

$\preceq$	$\perp$	0	$n>0$
$\perp$	$\perp$	tt	~~tt~~ $\perp$
m	$\perp$	tt	tt if $n \le m$ ff if $n > m$

Thus $\preceq$ is monotone

Suppose $(x,y) \sqsubseteq (x',y')$ and $\preceq(x,y) \neq \perp$

Case $x=0$, $y=\perp$. Then $\preceq(x,y) = tt$

Then $x'=0$, $y'=\perp$ or $y'=m \in \mathbb{N}$

In both cases $\preceq(x',y') = tt$ so $\checkmark$

Case $x=0$, $y=m \in \mathbb{N}$

Then $x'=x$, $y'=y$ so $\checkmark$

Case $x=n>0$, $y=m \in \mathbb{N}$

Then $x'=x$, $y'=y$ so $\checkmark$ $\boxtimes$

(8)

6a $[\![\mathbf{fix}(\mathbf{fn}\ f : \tau \to \tau .\ f)]\!] = \mathit{fix}(\lambda d \in [\![\tau \to \tau]\!] .\ d)$

$$= \bigsqcup_{n \in \mathbb{N}} Id^n(\bot) \quad = \bot_{[\![\tau]\!] \to [\![\tau]\!]}$$

$Id = \lambda d \in [\![\tau \to \tau]\!] .\ d$

$[\![\mathbf{fn}\ y : \tau (\mathbf{fix}\ x : \tau .\ x)]\!] = \lambda d \in [\![\tau]\!] .\ \mathit{fix}(\lambda e \in [\![\tau]\!] .\ e)$

$$\mathit{fix}(\lambda e \in [\![\tau]\!] .\ e) = \bigsqcup_{n \in \mathbb{N}} Id^n(\bot) = \bot$$

so we have as interpretation:

$$\lambda d \in [\![\tau]\!] .\ \bot_{[\![\tau]\!]}$$

This is the same as $\bot_{[\![\tau]\!] \to [\![\tau]\!]}$ ☒

6b ~~...~~

We show that for every valuation ρ of Γ $[\![S]\!](\rho) = [\![T]\!](\rho)$, assuming that S and T are well formed of some type τ in Γ.

Cases $[\![p]\!](\rho) = \bot$. Then $[\![S]\!](\rho) = [\![T]\!](\rho) = \bot$

$[\![p]\!](\rho) = tt$ $[\![q]\!](\rho) = \bot$ then $[\![S]\!](\rho) = \bot$, $[\![T]\!](\rho) = \bot$

—— $[\![q]\!](\rho) = tt$ then $[\![S]\!](\rho) = [\![K]\!](\rho)$

$[\![T]\!](\rho) = [\![K]\!](\rho)$

—— $[\![q]\!](\rho) = ff$ then $[\![S]\!](\rho) = [\![L]\!]_\rho$ $[\![T]\!](\rho) = [\![L]\!](\rho)$

$[\![p]\!](\rho) = ff$ $[\![r]\!](\rho) = \bot$ then $[\![S]\!](\rho) = \bot$ $[\![T]\!](\rho) = \bot$

—— $[\![r]\!](\rho) = tt$. then $[\![S]\!](\rho) = [\![K]\!](\rho)$, $[\![T]\!](\rho) = [\![K]\!](\rho)$

—— $[\![r]\!](\rho) = ff$, then $[\![S]\!](\rho) = [\![L]\!](\rho)$

$[\![T]\!](\rho) = [\![L]\!](\rho)$ ☒

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$$[\![(\underline{\text{if}}\ M_1\ \underline{\text{then}}\ M_2\ \underline{\text{else}}\ M_3) [N/x]]\!] \rho$$

$$= \begin{cases} \bot & \text{if } [\![M_1 [N/x]]\!](\rho) = \bot \\ [\![M_2 [N/x]]\!](\rho) & \text{if } [\![M_1 [N/x]]\!](\rho) = tt \\ [\![M_3 [N/x]]\!](\rho) & \text{if } [\![M_1 [N/x]]\!](\rho) = ff \end{cases}$$

$$\overset{IH}{=} \begin{cases} \bot & \text{if } [\![M_1]\!](\rho[x \mapsto [\![N]\!](\rho)) = \bot \\ [\![M_2]\!](\rho[x \mapsto [\![N]\!](\rho)) & \text{if } [\![M_1]\!](\rho[x \mapsto [\![N]\!](\rho)) = tt \\ [\![M_3]\!](\rho[x \mapsto [\![N]\!](\rho)) & \text{if } [\![M_1]\!](\rho[x \mapsto [\![N]\!](\rho)) = ff \end{cases}$$

$$= [\![\underline{\text{if}}\ M_1\ \underline{\text{then}}\ M_2\ \underline{\text{else}}\ M_3]\!] (\rho[x \mapsto [\![N]\!](\rho)])$$

$$[\![(\text{fn}\ x{:}\tau .\ M)[N/y]]\!](\rho) = [\![\text{fn}\ x{:}\tau .\ M[N/y]]\!](\rho)$$

We may assume that $x \notin FV(N)$

$$= \lambda d \in [\![\tau]\!].\ [\![M [N/y]]\!](\rho[x \mapsto d])$$

$$\overset{IH}{=} \lambda d \in [\![\tau]\!].\ [\![M]\!](\rho[x \mapsto d, y \mapsto [\![N]\!](\rho[x \mapsto d])])$$

because $x \notin FV(N)$

$$= \lambda d \in [\![\tau]\!].\ [\![M]\!](\rho[y \mapsto [\![N]\!](\rho), x \mapsto d])$$

$$= [\![\text{fn}\ x{:}\tau .\ M]\!](\rho[y \mapsto [\![N]\!](\rho)])$$

$\boxtimes$

8 To Prove:

$$\text{If } \underbrace{d_2 \sqsubseteq d_1}_{1},\ \underbrace{d_1 \lhd_\tau M_1}_{2} \text{ and } \overbrace{\forall V (M_1 \Downarrow_\tau V \Rightarrow M_2 \Downarrow_\tau V)}^{3},$$

then $d_2 \lhd_\tau M_2$

Induction on τ.

- $\tau = nat$

  The only case where $d_2 \sqsubseteq d_1$ and $d_2 \neq d_1$ is when $d_2 = \bot$, $d_1 \in \mathbb{N}$.

  But $\bot \lhd_\tau M_2$ always (by 7.2.1 (i))

- $\tau = bool$

  Similar to $\tau = nat$ case.

- $\tau = \tau_1 \to \tau_2$

  Suppose 1, 2, 3 for $\tau = \tau_1 \to \tau_2$

  Let e, N be such that $e \lhd_{\tau_1} N$ TP: $d_2(e) \lhd_{\tau_2} M_2\, N$

  By 2 we know $d_1(e) \lhd_{\tau_2} M_1\, N$ (using def of $\lhd_{\tau_1 \to \tau_2}$) $\ominus$

  We also have: $d_2(e) \sqsubseteq d_1(e)$ (from 1 $d_2 \sqsubseteq d_1$) $\oplus$

  $$\forall V (M_1 N \Downarrow_{\tau_2} V \Rightarrow M_2 N \Downarrow_{\tau_2} V) \quad \circledast$$

  So by IH for τ_2, ~~[illegible]~~ using $\ominus$ $\oplus$ and $\circledast$, we conclude

  $d_2(e) \lhd_{\tau_2} M_2\, N$

  NB $\circledast$ follows from 3: $\forall V (M_1 \Downarrow_{\tau_1 \to \tau_2} V \Rightarrow M_2 \Downarrow_{\tau_1 \to \tau_2} V)$

  Let V be a value with $M_1 N \Downarrow_{\tau_2} V$,

  Then $$\frac{M_1 \Downarrow \text{fn } x : \tau_1 . P \qquad P[N/x] \Downarrow_{\tau_2} V}{M_1 N \Downarrow_{\tau_2} V}$$

  So $$\frac{M_2 \Downarrow \text{fn } x : \tau_1 . P \qquad P[N/x] \Downarrow_{\tau_2} V}{M_2 N \Downarrow_{\tau_2} V}$$

$\boxtimes$

9a Suppose $|D|$ is finite, $|D| \geq 2$, say $|D| = n$

Then $[D \to D]$ contains $\left.\begin{array}{l} \lambda d \in D.\, d \\ \lambda d \in D.\, e_1 \\ \vdots \\ \lambda d \in D.\, e_n \end{array}\right\} (*)$

where $D = \{e_1, \dots, e_n\}$

NB the functions in $(*)$ are all continuous and are all different, so $|[D \to D]| \geq n+1 > |D|$.

9b A λ-model is a triple (D, F, G) with D a complete lattice

and $\begin{tikzcd} {[D \to D]} \arrow[r, bend left, "G"] & D \arrow[l, bend left, "F"] \end{tikzcd}$ such that $F \circ G = id_{[D \to D]}$

Then G is an injection: $G(g) = G(f) \Rightarrow F(G(g)) = F(G(f))$

$\Rightarrow g = f$.

so $|[D \to D]| \leq |D|$ for finite D,

contradicting 9a. So D cannot be finite with $|D| \geq 2$.

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10 • $BT(M_1) = BT(\lambda y.\, y M_1) = \begin{array}{c} \lambda y.\, y \\ | \\ BT(M_1) \end{array}$ So $BT(M_1) = \begin{array}{c} \lambda y.\, y \\ | \\ \lambda y'.\, y' \\ | \\ \lambda y''.\, y'' \\ | \\ \vdots \end{array}$

• $BT(M_2) = BT(\lambda y.\, y(M_2 y)) = \begin{array}{c} \lambda y.\, y \\ | \\ BT(M_2 y) \end{array}$

$BT(M_2 y) = BT(y(M_2 y)) = \begin{array}{c} y \\ | \\ BT(M_2 y) \end{array}$

so $BT(M_2) = \begin{array}{c} \lambda y.\, y \\ | \\ y \\ | \\ y \\ \vdots \end{array}$

• $BT(M_3) = BT(\lambda y.\, y M_3) = BT(M_1)$

So $D_A \models M_1 = M_3$ $\quad D_A \not\models M_1 = M_2$

11 a $G \circ F \not\sqsubseteq id_D$, $G \circ F \not\sqsupseteq id_D$

For $d \in D_A$, $G(F(d)) = G(\lambda e.\ F(d)(e))$

$$= \{(\beta, b) \mid b \in F(d)(\beta)\}$$

$$= \underline{\{(\beta, b) \mid \exists \beta' \subseteq \beta\ (\beta', b) \in d\}}$$

$d \not\subseteq G(F(d))$ in general, because
if d contains an element a from A,
then $a \notin G(F(d))$

$d \not\supseteq G(F(d))$ in general, because
if $(\gamma, c) \in d$, then $(\beta, c) \in G(F(d))$
for all finite $\beta \supseteq \gamma$

11 b Let $X \subseteq D_A$

To prove: $F(\bigsqcup X)(y) = \bigsqcup_{x \in X} (F(x)(y))$ for all y.

Here $\bigsqcup X$ is just $\bigcup X$, so

$$F(\bigcup X)(y) = \{b \mid \exists \beta \subseteq y\ (\beta, b) \in \bigcup X\}$$

$$= \bigcup_{x \in X} \{b \mid \exists \beta \subseteq y\ (\beta, b) \in x\}.$$

$$= \bigcup_{x \in X} F(x)(y). \quad \boxtimes$$

In case you don't see this equality, prove $\subseteq$ and $\supseteq$

$\subseteq$: If $\exists \beta \subseteq y\, (\beta, b) \in \bigcup X$, then $\exists \beta \subseteq y\, (\beta, b) \in x$ for some $x \in X$
so $b \in \bigcup_{x \in X} \{b \mid \exists \beta \subseteq y\, (\beta, b) \in x\}$

$\supseteq$: If, for some $x \in X$, $\exists \beta \subseteq y\, (\beta, b) \in x$, then $\exists \beta \subseteq y\ (\beta, b) \in \bigcup X$,
so $b \in \{b \mid \exists \beta \subseteq y\, (\beta, b) \in \bigcup X\}$.