RADBOUD UNIVERSITY NIJMEGEN

Science Faculty

Exam **Semantics and Domain Theory** July 2, 2014, 8.30 – 11.30
The maximum number of points per question is given in the margin. (Maximum 100 points in total.) DENS refers to the course notes of Winskell.
NB: you can use all your notes and course material. When dealing with well-known functions and chains that we have seen in the course, you don't have to show (again) that they are monotone/continuous etc.

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1. We define the program

$$S := \textbf{while}\ x > 0\ \textbf{do}\ (y := y + x; x := x - 1)$$

Define $w_n : \text{State} \rightharpoonup \text{State}$ as follows: $w_0(s) = \bot$ and for $n > 0$,

$$w_n(s) := \begin{cases} s & \text{if } s(x) \leq 0 \\ s[x \mapsto 0, y \mapsto s(y) + \frac{s(x)\,(s(x)+1)}{2}] & \text{if } 0 < s(x) < n \\ \bot & \text{if } n \leq s(x) \end{cases}$$

(10) (a) Give the functional F that we need in order to compute the denotational semantics of S and show that $F(w_n) = w_{n+1}$

(10) (b) Give $w_\infty := \sqcup_{n \in \mathbb{N}} w_n$ and show that w_∞ is a fixed-point of F.

(10) 2. Let f and g be monotone functions from the cpo $(D, \sqsubseteq)$ to $(D, \sqsubseteq)$. Assume that (i) $f(\bot) = g(\bot)$ and (ii) $f \circ g = g \circ f$.
Prove that $\text{fix}\, f = \text{fix}\, g$.
(NB. fix is the well-known Tarksi *least-fixed point* of f that we have seen in the course.)

Continue on other side

3. We define the "parallel implication' function pimp : $\mathbb{B}_\bot \times \mathbb{B}_\bot \to \mathbb{B}_\bot$ as follows (the first argument is in horizontal position, the second argument in vertical position):

pimp	$\bot$	**tt**	**ff**
$\bot$	$\bot$	$\bot$	**tt**
tt	**tt**	**tt**	**tt**
ff	$\bot$	**ff**	**tt**

(5) (a) Prove that pimp is monotone.

(5) (b) Prove that pimp is continuous.

(10) (c) Prove that pimp is not definable in PCF.

(5) (d) Change one entry in the table of the definition of pimp and show that your adapted function pimp$'$ is definable in PCF.

4. Consider the following PCF terms

$$\begin{aligned} P_1 &:= \mathbf{fix}\,(\mathbf{fn}\ f : \mathbf{nat} \to \mathbf{nat}.\mathbf{fn}\ x : \mathbf{nat}.\,\mathbf{if\ zero}\,x\ \mathbf{then}\ x\ \mathbf{else}\ f(\mathbf{pred}(x))) \\ P_2 &:= \mathbf{fn}\ x : \mathbf{nat}.\,0 \\ P_3 &:= \mathbf{fn}\ x : \mathbf{nat}.\,\mathbf{if\ zero}\,x\ \mathbf{then}\ x\ \mathbf{else}\ 0 \end{aligned}$$

(10) (a) Give a term V such that $P_1(\mathbf{succ}(0)) \Downarrow_{\mathbf{nat}} V$ and give a derivation of $P_1(\mathbf{succ}(0)) \Downarrow_{\mathbf{nat}} V$.

(10) (b) Give $[\![P_1]\!]$ and show how you have obtained that answer. (You don't have to give a full computation.)

(10) (c) For which i, j do we have $[\![P_i]\!] \neq [\![P_j]\!]$?

(8) 5. Suppose that the λ-terms M_2, M_2 and M_3 satisfy

$$\begin{aligned} M_1 &=_\beta \lambda x.x\,M_2\,(\lambda y.x\,M_1) \\ M_2 &=_\beta \lambda x.x\,M_1\,(\lambda y.y\,M_3) \\ M_3 &=_\beta \lambda x.x\,M_2\,(\lambda y.x\,M_3) \end{aligned}$$

For which i, j (with $i \neq j$) is it the case that $D_A \models M_i = M_j$? Prove your answer.

(7) 6. Let M and N be λ-terms and assume that M does not have a head-normal form, while N does have a head-normal form.
Show that in the model D_A, $[\![\lambda x.x\,M]\!] \subseteq [\![\lambda x.x\,N]\!]$.

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