Semantics and Domain theory

Exercises 1

At the lecture, we gave a denotational semantics for the language \( \mathcal{L} \) given by the grammar

\[
\begin{align*}
    b &::= 0 | 1 \\
n &::= b | n b
\end{align*}
\]

NB 0 and 1 are symbols, not the numbers.

The semantics is given by the model \( \mathbb{N} \), the natural numbers, and the interpretation

\[
\begin{align*}
    [0] &:= 0 \\
    [1] &:= 1 \\
    [n b] &:= 2 \cdot [n] + [b]
\end{align*}
\]

In the lecture, we have recursively defined the operation \( P(n) \), which prefixes a binary numeral \( n \) with a leading 0 as follows.

\[
\begin{align*}
P(0) &:= 0 0 \\
P(1) &:= 0 1 \\
P(n b) &:= P(n) b
\end{align*}
\]

We have given an operational semantics \( \rightarrow \) via the rules

\[
\begin{align*}
0 &\rightarrow 0 0 \\
1 &\rightarrow 0 1 \\
\frac{n \rightarrow m}{n b \rightarrow m b}
\end{align*}
\]

Exercises:

1. Define the operation \( S(n) \), which computes the binary numeral which is the successor of \( n \).

2. (a) Give an operational semantics for \( S(n) \), in the form of a relation \( n \rightarrow m \) such that \( S(n) = m \) iff \( n \rightarrow m \)

(b) Prove that \( S(n) = m \) iff \( n \rightarrow m \)

3. Prove \( [S(n)] = [n] + 1 \) for all \( n \).

4. (a) Compute the denotational semantics of \( S_1 := x := x + 1 ; \ y := x + x \)

(b) Compute the denotational semantics of \( S_2 := \) if \( x > 0 \) then \( x := 1 \) else \( x := -1 \)

NB Your answer should be a "state transformers", i.e. an element of \( \text{State} \rightarrow \text{State} \), the set of partial functions from \( \text{State} \) to \( \text{State} \). For us a state is a function from locations (variables) to integers, \( r : \mathbb{L} \rightarrow \mathbb{Z} \).