Recall the following terms in untyped λ-calculus

- \( c_n \) denotes the \( n \)-th Church numeral, so in particular \( c_0 = \lambda f.x.x \), \( c_1 = \lambda f.x.f \) and in general \( c_n = \lambda f.f^n(x) \).

- \( K = \lambda x y.x \), \( I = \lambda x.x \), \( \Omega = (\lambda x.x)(\lambda x.x) \).

1. (a) Prove that the equation \( c_0 = c_1 \) is inconsistent in untyped \( \lambda \) calculus. (That is: show that, if you assume \( c_0 = c_1 \), then you can prove \( M = N \) for all terms \( M, N \).)

(b) Prove that \( c_0 = c_{n+1} \) is inconsistent for any \( n \in \mathbb{N} \).

(c) Prove that \( c_n = c_m \) is inconsistent for \( n, m \in \mathbb{N} \) with \( n \neq m \).

2. The applicative structure \((M, \cdot)\), with \( M = \mathbb{N} \) (the natural numbers) and \( \cdot = * \) (multiplication) cannot be made into a (consistent) model of the untyped \( \lambda \)-calculus. We prove this in the following steps:

   Consider \((\mathbb{N}, \ast)\) and assume that there is an interpretation \([\_\_]\) satisfying the definition that we have given in the lecture (see Definition 59 of Berline). Now:

   (a) Show that \([K I] = [I K]\).

   (b) Conclude that \( d = e \) for all \( d, e \in \mathbb{N} \).

So: all elements are equal in the model.

3. Prove that the theory that equates all λ-terms that don’t have a normal form is inconsistent by showing that the following equation is inconsistent in untyped λ calculus:

\[
\lambda x y.x y \Omega = \lambda x y.y x \Omega.
\]