Semantics and Domain theory

Exercises 1

At the lecture, we gave a denotational semantics for the language $\mathcal{L}$ given by the grammar

$$\begin{array}{rcl} b : \mathsf{bit} & ::= & \mathbf{0} \mid \mathbf{1} \\ n : \mathsf{bin} & ::= & b \mid n\, b \end{array}$$

NB **0** and **1** are symbols, not the numbers.

The semantics is given by the model $\mathbb{N}$, the natural numbers, and the interpretation

$$\begin{array}{rcl} [\![\mathbf{0}]\!] & := & 0 \\ [\![\mathbf{1}]\!] & := & 1 \\ [\![n\, b]\!] & := & 2 * [\![n]\!] + [\![b]\!] \end{array}$$

In the lecture, we have recursively defined the operation $P(n)$, which prefixes a binary numeral n with a leading **0** as follows.

$$\begin{array}{rcl} P(\mathbf{0}) & := & \mathbf{0\,0} \\ P(\mathbf{1}) & := & \mathbf{0\,1} \\ P(n\, b) & := & P(n)\, b \end{array}$$

We have given an operational semantics $\stackrel{P}{\Longrightarrow}$ via the rules

$$\frac{}{\mathbf{0} \stackrel{P}{\Longrightarrow} \mathbf{0\,0}} \qquad \frac{}{\mathbf{1} \stackrel{P}{\Longrightarrow} \mathbf{0\,1}} \qquad \frac{n \stackrel{P}{\Longrightarrow} m}{n\, b \stackrel{P}{\Longrightarrow} m\, b}$$

Exercises:

1. Define the operation $S(n)$, which computes the binary numeral which is the successor of n.
   Answer: ....

   $$\begin{array}{rcl} S(\mathbf{0}) & := & \mathbf{1} \\ S(\mathbf{1}) & := & \mathbf{10} \\ S(n\,\mathbf{0}) & := & n\mathbf{1} \\ S(n\,\mathbf{1}) & := & S(n)\mathbf{0} \end{array}$$

   End answer...

2. (a) Give an operational semantics for $S(n)$, in the form of a relation $n \stackrel{S}{\Longrightarrow} m$ such that $S(n) = m$ iff $n \stackrel{S}{\Longrightarrow} m$
   Answer: ....

   $$\frac{}{\mathbf{0} \stackrel{S}{\Longrightarrow} \mathbf{1}} \qquad \frac{}{\mathbf{1} \stackrel{S}{\Longrightarrow} \mathbf{1\,0}} \qquad \frac{}{n\,\mathbf{0} \stackrel{S}{\Longrightarrow} n\,\mathbf{1}} \qquad \frac{n \stackrel{S}{\Longrightarrow} m}{n\,\mathbf{1} \stackrel{S}{\Longrightarrow} m\,\mathbf{0}}$$

   End answer...

(b) Prove that $S(n) = m$ iff $n \overset{S}{\Longrightarrow} m$

Answer: . . . .

We prove

(1) (⇒): $n \overset{S}{\Longrightarrow} S(n)$ by induction on n.

(2) (⇐): if $n \overset{S}{\Longrightarrow} m$, then $S(n) = m$ by induction on the derivation of $n \overset{S}{\Longrightarrow} m$.

(1)

- *cases $\mathbf{0}$ and $\mathbf{1}$: immediate*
- *case $m\mathbf{0}$: $m\mathbf{0} \overset{S}{\Longrightarrow} m\mathbf{1} = S(m)$, done*
- *case $m\mathbf{1}$: IH: $m \overset{S}{\Longrightarrow} S(m)$, so by the fourth rule for $\overset{S}{\Longrightarrow}$, we conclude that $m\mathbf{1} \overset{S}{\Longrightarrow} S(m)\mathbf{0}$. We have $S(m\mathbf{1}) = S(m)\mathbf{0}$, so done.*

(2)

- *the cases for the first two rules, for $\mathbf{0}$ and $\mathbf{1}$ are immediate.*
- *case of the 3d rule: $m\mathbf{0} \overset{S}{\Longrightarrow} m\mathbf{1}$: $S(m\mathbf{0}) = m\mathbf{1}$, so done.*
- *case of the 4th rule:*

$$\frac{m \overset{S}{\Longrightarrow} p}{m\,\mathbf{1} \overset{S}{\Longrightarrow} p\,\mathbf{0}}$$

IH: $S(m) = p$. To prove: $S(m\mathbf{1}) = p\mathbf{0}$. We have $S(m\mathbf{1}) = S(m)\mathbf{0} \overset{IH}{=} p\mathbf{0}$, so done.

End answer . . . .

3. Prove $[\![S(n)]\!] = [\![n]\!] + 1$ for all n.

Answer: . . . .

By induction on n

- $[\![S(\mathbf{0})]\!] = [\![\mathbf{1}]\!] = 1 = [\![\mathbf{0}]\!] + 1$.
- $[\![S(\mathbf{1})]\!] = [\![\mathbf{10}]\!] = 2 = [\![\mathbf{1}]\!] + 1$.
- $[\![S(m\mathbf{0})]\!] = [\![m\mathbf{1}]\!] = 2[\![m]\!] + 1 = 2[\![m]\!] + 0 + 1 = [\![m\mathbf{0}]\!] + 1$.
- $[\![S(m\mathbf{1})]\!] = [\![S(m)\mathbf{0}]\!] = 2[\![S(m)]\!] + 0 \overset{IH}{=} 2([\![m]\!] + 1) = 2[\![m]\!] + 1 + 1 = [\![m\mathbf{1}]\!] + 1$.

End answer . . . .

4. (a) Compute the denotational semantics of $S_1 :\equiv x := x + 1;\ y := x + x$

Answer: . . . .

$$\begin{aligned}
[\![S_1]\!] &= \boldsymbol{\lambda}\, s : \mathsf{State}.[\![y := x + x]\!]([\![x := x + 1]\!](s)) \\
&= \boldsymbol{\lambda}\, s : \mathsf{State}.[\![y := x + x]\!](s[x \mapsto s(x) + 1]) \\
&= \boldsymbol{\lambda}\, s : \mathsf{State}.s[x \mapsto s(x) + 1, y \mapsto s(x) + 1 + s(x) + 1] \\
&= \boldsymbol{\lambda}\, s : \mathsf{State}.s[x \mapsto s(x) + 1, y \mapsto 2s(x) + 2]
\end{aligned}$$

End answer . . . .

(b) Compute the denotational semantics of $S_2 :\equiv$ if $x > 0$ then $x := 1$ else $x := -1$

Answer: ....

$$[\![S_2]\!] = \boldsymbol{\lambda}\, s : \mathsf{State}. \begin{cases} s[x \mapsto 1] & \textit{if } s(x) > 0 \\ s[x \mapsto -1] & \textit{if } s(x) \leq 0 \end{cases}$$

End answer ....

NB Your answer should be a "state transformers", i.e. an element of $\mathsf{State} \rightharpoonup \mathsf{State}$, the set of partial functions from State to State. For us a state is a function from locations (variables) to integers, $r : \mathbb{L} \to \mathbb{Z}$.