Semantics and Domain theory
Exercises 2

NB. We write $\text{State} \rightarrow \text{State}$ for the set of partial functions from $\text{State}$ to $\text{State}$.

1. Define the denotational semantics of $\text{repeat } P \text{ until } b$ as a fixed point of a function $g : (\text{State} \rightarrow \text{State}) \rightarrow (\text{State} \rightarrow \text{State})$.
   (NB this program executes statement $P$ and then checks the boolean $b$; if $b$ holds, execution stops, if $b$ doesn’t hold, it iterates.)

   \textbf{Answer:} \hspace{1cm} \hspace{1cm}
   We use the idea that $\text{repeat } P \text{ until } b$ has the same meaning as
   \[ P; \text{ if } b \text{ then skip else (repeat } P \text{ until } b). \]

   We define $[\text{repeat } P \text{ until } b]$ as the fixed point of $g : (\text{State} \rightarrow \text{State}) \rightarrow (\text{State} \rightarrow \text{State})$, where
   \[ g := \lambda w : \text{State} \rightarrow \text{State}. \lambda s : \text{State}. \text{IF}([b]([P](s)), [P](s), w([P](s))). \]
   where $\text{IF} : \mathbb{B} \times \text{State} \times \text{State} \rightarrow \text{State}$ is defined by $\text{IF}(t, s_1, s_2) = s_1$ and $\text{IF}(f, s_1, s_2) = s_2$.

   Alternatively:
   \[ g := \lambda w : \text{State} \rightarrow \text{State}. \text{If}([b], \text{Id}, w) \circ [P], \]
   where $\text{If} : (\text{State} \rightarrow \mathbb{B}) \times (\text{State} \rightarrow \text{State}) \times (\text{State} \rightarrow \text{State}) \rightarrow (\text{State} \rightarrow \text{State})$ is defined by $\text{If}(f, w_1, w_2) = \lambda s : \text{State}. \text{If}(f(s), w_1(s), w_2(s))$.

   \textbf{End answer} \hspace{1cm} \hspace{1cm}

2. [Exercise 4.2 of Nielsen & Nielsen] Consider the statement
   \[ S := \text{ while } x \neq 0 \text{ do } x := x - 1 \]

   (a) Determine the functional $F : (\text{State} \rightarrow \text{State}) \rightarrow (\text{State} \rightarrow \text{State})$ associated with this statement. (The $F$ we need to take the fixed point of to determine the semantics of $S$.)

   \textbf{Answer:} \hspace{1cm} \hspace{1cm}
   \[ F(w)(s) := \begin{cases} 
   s & \text{if } s(x) = 0 \\
   w(s[x \mapsto s(x) - 1]) & \text{if } s(x) \neq 0 
   \end{cases} \]

   \textbf{End answer} \hspace{1cm} \hspace{1cm}

   (b) Determine for each of the following partial functions $g : \text{State} \rightarrow \text{State}$ whether it is a fixed point of $F$.

   - $g_1(s) := \uparrow$ for all $s \in \text{State}$
   - $g_2(s) := \{ \begin{cases} 
   s[x \mapsto 0] & \text{if } s(x) \geq 0 \\
   \uparrow & \text{if } s(x) < 0 
   \end{cases} \}
   - $g_3(s) := \{ \begin{cases} 
   s[x \mapsto 0] & \text{if } s(x) \geq 0 \\
   s & \text{if } s(x) < 0 
   \end{cases} \}
   - $g_4(s) := s[x \mapsto 0]$ for all $s \in \text{State}$
   - $g_5(s) := s$ for all $s \in \text{State}$
3. Consider the function $f$ as defined in the lecture, for our notion of State.

(a) Prove $f(w_n) = w_{n+1}$ for $w_n : \text{State} \rightarrow \text{State}$ as defined in the lecture, for our notion of State.

**Answer:** .................................................................

For $n \geq 1$, $w_n$ has the following definition

$$w_n(s) := \begin{cases} 
  s & \text{if } s(x) \leq 0 \\
  w_n(s[x \mapsto s(x) - 1, y \mapsto s(x) * s(y)]) & \text{if } s(x) > 0 
\end{cases}$$

We have

$$f(w_n)(s) = \begin{cases} 
  s & \text{if } s(x) \leq 0 \\
  w_n(s[x \mapsto s(x) - 1, y \mapsto s(x) * s(y)]) & \text{if } s(x) > 0 
\end{cases}$$

$$= \begin{cases} 
  s & \text{if } s(x) \leq 0 \\
  s[x \mapsto s(x) - 1, y \mapsto s(x) * s(y)] & \text{if } s(x) = 1 \\
  s[x \mapsto 0, y \mapsto (s(x) - 1)! * s(x) * s(y)] & \text{if } 0 < s(x) - 1 < n \\
  \uparrow & \text{if } s(x) - 1 \geq n \\
  s & \text{if } s(x) \leq 0 \\
  s[x \mapsto 0, y \mapsto s(x)! * s(y)] & \text{if } 0 < s(x) < n + 1 \\
  \uparrow & \text{if } s(x) \geq n + 1 \\
  w_{n+1}(s) & \text{if } s(x) \leq 0 \\
  w_n(s[x \mapsto s(x) - 1, y \mapsto s(x) * s(y)]) & \text{if } s(x) > 0 
\end{cases}$$

**End answer** .................................................................

(b) Prove $f(w_\infty) = w_\infty$ for $w_\infty : \text{State} \rightarrow \text{State}$ as defined in the lecture, for our notion of State.

**Answer:** .................................................................

$w_\infty$ has the following definition

$$w_\infty(s) := \begin{cases} 
  s & \text{if } s(x) \leq 0 \\
  s[x \mapsto 0, y \mapsto s(x)! * s(y)] & \text{if } s(x) > 0 
\end{cases}$$

**End answer** .................................................................
We have

\[
f(w_\infty)(s) = \begin{cases} 
  s & \text{if } s(x) \leq 0 \\
  w_\infty(s[x \mapsto s(x) - 1, y \mapsto s(x) \ast s(y)]) & \text{if } s(x) > 0 
\end{cases}
\]

\[
= \begin{cases} 
  s & \text{if } s(x) \leq 0 \\
  s[x \mapsto s(x) - 1, y \mapsto s(x) \ast s(y)] & \text{if } s(x) = 1 \\
  s[x \mapsto 0, y \mapsto (s(x) - 1)! \ast s(x) \ast s(y)] & \text{if } s(x) > 1 
\end{cases}
\]

\[
= \begin{cases} 
  s & \text{if } s(x) \leq 0 \\
  s[x \mapsto 0, y \mapsto (s(x) - 1)! \ast s(x) \ast s(y)] & \text{if } s(x) > 0 
\end{cases}
\]

\[
= w_\infty
\]

End answer

(c) Show implication (3) on page 19, that is: for all \(w\),

\[
w = f(w) \Rightarrow w_\infty \subseteq w.
\]

**Answer:** Suppose \(w = f(w)\). We show that for all \(s \in \text{State}\), \(w_\infty(s) = w(s)\). We distinguish two cases.

- \(s(x) \leq 0\). Then \(w(s) = f(w)(s) = s = w_\infty(s)\), so done.
- \(s(x) \geq 0\). We do induction on \(n := s(x)\).
  - \(n = 0\). Done.
  - \(IH\): for states \(s\) with \(s(x) = n\) we know \(w(s) = w_\infty(s)\). Now suppose \(s\) is a state with \(s(x) = n + 1\). Then

\[
w(s) = f(w)(s) = \begin{cases} 
  w(s[x \mapsto s(x) - 1, y \mapsto s(x) \ast s(y)]) & \text{if } s(x) \leq 0 \\
  \overset{IH}{=} w_\infty(s[x \mapsto s(x) - 1, y \mapsto s(x) \ast s(y)]) 
\end{cases}
\]

\[
= s[x \mapsto 0, y \mapsto (s(x) - 1)! \ast s(x) \ast s(y)]
\]

\[
= w_\infty(s).
\]

End answer

(d) Prove that \(\forall s \in \text{State} \exists n[f^n(\bot)(s)] = f^{n+1}(\bot)(s)\).

**Answer:** Let \(s \in \text{State}\). As \(w_n = f^n(\bot)\), we are done if we prove \(\exists n[w_n(s) = w_{n+1}(s)]\), where \(w_n\) has been defined above. In case \(s(x) \leq 0\), take \(n := 1\). In case \(s(x) > 0\): take \(n := s(x) + 1\). Then \(s(x) < n\), so \(w_n(s) = s[x \mapsto 0, y \mapsto (s(x) - 1)! \ast s(y)] = w_{n+1}(s)\).

End answer

4. [Extra exercise to possibly think about] Define a denotational semantics for the statement for \(x := e_1\ to e_2\ do P\):

(a) First with \(e_1, e_2\) fixed numbers in \(\mathbb{Z}\), say \(n\) and \(m\).

(b) Discuss some of the choices and problems with giving the general semantics, where \(e_1\) and \(e_2\) are arbitrary expressions.

What semantics would you give to for \(x := 1 \ to x + 1\ do \text{skip}\)? And to for \(x := 1 \ to 3\ do x := x - 1\)?