Semantics and Domain theory

Exercises 3

NB. A domain is a cpo with a $\bot$-element.

1. Prove that the set of partial functions from X to Y, $X \rightharpoonup Y$, forms a domain, with the definitions of ordering and lub given on slide 24 of the lecture notes (or the equivalent definitions given at the lecture).
   Answer: ....
   For $f, g \in X \rightharpoonup Y$, we have

   $$f \sqsubseteq g \text{ iff } \mathrm{dom}(f) \subseteq \mathrm{dom}(g) \wedge \forall x \in \mathrm{dom}(f).\, f(x) = g(x),$$

   or alternatively $\forall x \in X.\, f(x)\downarrow \Rightarrow f(x) = g(x)$.

   We check the properties of a domain.

   - $\bot \in X \rightharpoonup Y$ defined by $\bot(x)\uparrow$ is the least element: $\mathrm{dom}(\bot) = \varnothing$, $\bot \sqsubseteq g$ for every $g \in X \rightharpoonup Y$ is immediate.
   - reflexivity of $\sqsubseteq$ is immediate.
   - transitivity: suppose $f \sqsubseteq g$ and $g \sqsubseteq h$ and let $x \in X$ with $f(x)\downarrow$. Then $f(x) = g(x)$ and so $g(x)\downarrow$, so $g(x) = h(x)$, so done.
   - symmetry: suppose $f \sqsubseteq g$ and $g \sqsubseteq f$. Then $\mathrm{dom}(f) = \mathrm{dom}(g)$ and for all $x \in X$, $f(x) = g(x)$, so $f = g$.
   - Let $(f_i)_{i\in\mathbb{N}}$ be a chain, so $f_0 \sqsubseteq f_1 \sqsubseteq f_2 \sqsubseteq \ldots$. Note that, if $x \in \mathrm{dom}(f_n)$, then, for all $m \geq n$: $x \in \mathrm{dom}(f_m)$ and $f_m(x) = f_n(x)$. Now, $\cup_{i\in\mathbb{N}} f_i$ is defined as te function f with

     $$f(x) := \begin{cases} f_n(x) & \text{if } x \in \mathrm{dom}(f_n) \text{ for some } n \\ \uparrow & \text{if } x \notin \mathrm{dom}(f_n) \text{ for all } n \end{cases}$$

     We need to prove the (lub1) and (lub2) property:
     (lub1) For f_i, if $f_i(x)\downarrow$, then $f(x)\downarrow$ and $f(x) = f_i(x)$, so $f_i \sqsubseteq f$ for all i.
     (lub2) Suppose $f_i \sqsubseteq g$ for all $i \in \mathbb{N}$. If $f(x)\downarrow$, then $f_n(x)\downarrow$ for some n, so $f_n(x) = g(x)$, so $f \sqsubseteq g$.

   End answer..

2. Which of the following partial orders is a domain? In each case, choose a proper definition of 'lub'; prove your answer. You don't have to prove it is a partial order, but please check for yourself that it is! (In case you haven't done so.)

   (a) $(\mathcal{P}(\mathbb{N}), \subseteq)$, where $\mathcal{P}(\mathbb{N})$ is the powerset of $\mathbb{N}$ (the set of all subsets of $\mathbb{N}$) and $\subseteq$ is the usual subset ordering.
   Answer: ....
   $\varnothing$ is the $\bot$-element. For $X_0 \subseteq X_1 \subseteq X_2 \subseteq \ldots$ a chain, we define $\sqcup_{i\in\mathbb{N}} X_i := \cup_{i\in\mathbb{N}} X_i$, that is: the union of all X_i. This is an upperbound of $(X_i)_{i\in\mathbb{N}}$, because $X_i \subseteq \cup_{i\in\mathbb{N}} X_i$. If Y is also an upperbound of $(X_i)_{i\in\mathbb{N}}$, then $X_i \subseteq Y$ for all i. Now, if $x \in \cup_{i\in\mathbb{N}} X_i$, then $x \in X_i$ for some i, so $x \in Y$. So $\cup_{i\in\mathbb{N}} X_i \subseteq Y$.
   End answer..

   (b) $(\mathcal{P}_{\text{fin}}(\mathbb{N}), \subseteq)$, where $\mathcal{P}_{\text{fin}}(\mathbb{N})$ is the set of finite subsets of $\mathbb{N}$ and $\subseteq$ is the usual subset ordering.
   Answer: ....

$(\mathcal{P}_{\text{fin}}(\mathbb{N}), \subseteq)$ is not a domain, because the chain $\{0\} \subseteq \{0,1\} \subseteq \{0,1,2\} \subseteq \ldots$ (written differently: $(\{0,\ldots,i\})_{\imath \in \mathbb{N}})$ has no lub in $\mathcal{P}_{\text{fin}}(\mathbb{N})$: If Y is an upperbound of this chain, $\{0,\ldots,i\} \subseteq Y$ for each $i \in \mathbb{N}$, so $\mathbb{N} \subseteq Y$, so Y is not finite.

End answer...

(c) $([0,1], \leq)$, where $[0,1]$ is the unit interval and $\leq$ is the usual ordering on the real numbers.

Answer: ....

0 is the $\bot$-element. If $(r_i)_{i\in\mathbb{N}}$ is a chain, its lub is $\lim_{i\to\infty} r_i$. We know from Calculus that this limit exists: every bounded increasing sequence in $\mathbb{R}$ has a limit and this limit is also the lub under the partial ordering $\leq$.

End answer...

(d) $([0,1] \cap \mathbb{Q}, \leq)$, where $\cap$ is the intersection and $\mathbb{Q}$ is the set of rational numbers.

Answer: ....

$([0,1] \cap \mathbb{Q}, \leq)$ is not a domain. Consider the decimal expansion of $r := \frac{1}{2}\sqrt{2}$ and define q_i as r up to the first i digits of its decimal expansion. Then $q_i \leq q_{i+1}$, so $(q_i)_{i\in\mathbb{N}}$ is a chain and its lub (in $\mathbb{R}$) is r. However, $r \notin \mathbb{Q}$, because r is not rational.

End answer...

(e) $(\Sigma^*, \sqsubseteq)$, where Σ^* is the set of words over the alphabet $\Sigma := \{a, b\}$ and $\sqsubseteq$ is the prefix ordering, defined by $w \sqsubseteq wv$ for all $w, v \in \Sigma^*$.

Answer: ....

$(\Sigma^*, \sqsubseteq)$ is not a domain. Consider the chain $(a^i)_{i\in\mathbb{N}}$ (words $a\ldots a$ of length i). Note that, if $v \sqsubseteq w$, then $|v| \leq |w|$ (where $|v|$ denotes the length of v). If w is the lub of $(a^i)_{i\in\mathbb{N}}$, then $|a^i| \leq |w|$ for all $i \in \mathbb{N}$, so w must have infinite length, whereas Σ^* contains finite words only.

End answer...

(f) $(\Sigma^* \cup \Sigma^\omega, \sqsubseteq)$, where Σ^ω is the set of infinite words over the alphabet $\Sigma := \{a, b\}$ and $\sqsubseteq$ is the prefix ordering, defined by $w \sqsubseteq wv$ for all $w \in \Sigma^*$, $v \in \Sigma^* \cup \Sigma^\omega$ and $v \sqsubseteq v$ for all $v \in \Sigma^* \cup \Sigma^\omega$.

Answer: ....

$(\Sigma^* \cup \Sigma^\omega, \sqsubseteq)$ is a domain with the empty word as $\bot$-element. If $(w_i)_{i\in\mathbb{N}}$ is a chain there are two possibilities:

- $(w_i)_{i\in\mathbb{N}}$ is eventually constant, say $w_N = w_{N+1} = \ldots$. Then $\sqcup_{i\in\mathbb{N}} w_i := w_N$, the lub of $(w_i)_{i\in\mathbb{N}}$.
- $(w_i)_{i\in\mathbb{N}}$ is not eventually constant, which means that the chain contains words of arbitrary length. Put in a formula: $\forall \ell \in \mathbb{N} \, \exists i \in \mathbb{N}. |w_i| > \ell$. Now, $\sqcup_{i\in\mathbb{N}} w_i$ is the infinite word w which has on position ℓ the letter that is on position ℓ in a word w_i from the chain for which we know that $|w_i| > \ell$. More precisely, if we indicate with $w(\ell)$ the letter at position ℓ in w:

  $$w(\ell) := w_i(\ell) \text{ if } w_i \text{ is a word in the chain with } |w_i| > \ell.$$

  It is immediate that w is an upperbound of $(w_i)_{i\in\mathbb{N}}$. It is also the least upperbound, because it is the only upperbound: an upperbound should satisfy the equation for w displayed above.

End answer...

3. Prove that the function $f_{b,C}$ in the definition of the denotational semantics of **while** B **do** C (slide 11) is continuous. When is $f_{b,C}$ strict?
Answer: ....
For b a boolean expression and C a command, $f_{b,C}$ is defined by

$$f_{b,C} := \boldsymbol{\lambda}\, w : \mathsf{State} \rightharpoonup \mathsf{State}.\ \boldsymbol{\lambda}\, s : \mathsf{State}.\ \mathrm{IF}(\llbracket b \rrbracket(s), w(\llbracket C \rrbracket(s)), s).$$

$f_{b,C}$ is strict iff $\llbracket b \rrbracket(s) = \mathrm{tt}$ for all $s \in \mathsf{State}$. (For example, when $b =$ **true** or $b = x \leq x$.)

- $f_{b,C}$ is monotonic: if $w_1 \sqsubseteq w_2$, then, for $s \in State$: (1) if $\llbracket b \rrbracket(s) = \mathrm{ff}$, then $f_{b,C}(w_1)(s) = s = f_{b,C}(w_2)(s)$; (2) if $\llbracket b \rrbracket(s) = \mathrm{tt}$, then $f_{b,C}(w_1)(s) = w_1(\llbracket C \rrbracket(s))$. If $w_1(\llbracket C \rrbracket(s)) \downarrow$, then $w_2(\llbracket C \rrbracket(s)) \downarrow$ and $w_2(\llbracket C \rrbracket(s)) = f_{b,C}(w_2)(s)$. So we have $f_{b,C}(w_1) \sqsubseteq f_{b,C}(w_2$.
- $f_{b,C}$ preserves $\sqcup$. Let $(w_i)_{i\in\mathbb{N}}$ be a chain in $\mathsf{State} \rightharpoonup \mathsf{State}$. We have to check that for all $s \in \mathsf{State}$, if $\mathrm{IF}(\llbracket b \rrbracket(s), (\sqcup_{i\in\mathbb{N}} w_i)(\llbracket C \rrbracket(s)), s) \downarrow$, then $\mathrm{IF}(\llbracket b \rrbracket(s), (\sqcup_{i\in\mathbb{N}} w_i)(\llbracket C \rrbracket(s)), s) = \sqcup_{i\in\mathbb{N}} \mathrm{IF}(\llbracket b \rrbracket(s), (w_i)(\llbracket C \rrbracket(s)), s)$. There are two cases: (1) if $\llbracket b \rrbracket(s) = \mathrm{ff}$, then $\mathrm{IF}(\llbracket b \rrbracket(s), (\sqcup_{i\in\mathbb{N}} w_i)(\llbracket C \rrbracket(s)), s) = s = \sqcup_{i\in\mathbb{N}} \mathrm{IF}(\llbracket b \rrbracket(s), w_i(\llbracket C \rrbracket(s)), s)$. (2) if $\llbracket b \rrbracket(s) = \mathrm{tt}$, then $\mathrm{IF}(\llbracket b \rrbracket(s), (\sqcup_{i\in\mathbb{N}} w_i)(\llbracket C \rrbracket(s)), s) = (\sqcup_{i\in\mathbb{N}} w_i)(\llbracket C \rrbracket(s))$. If $(\sqcup_{i\in\mathbb{N}} w_i)(\llbracket C \rrbracket(s)) \downarrow$, then $(\sqcup_{i\in\mathbb{N}} w_i)(\llbracket C \rrbracket(s)) = w_n(\llbracket C \rrbracket(s))$ for some n, so $\mathrm{IF}(\llbracket b \rrbracket(s), (\sqcup_{i\in\mathbb{N}} w_i)(\llbracket C \rrbracket(s)), s) = \sqcup_{i\in\mathbb{N}} \mathrm{IF}(\llbracket b \rrbracket(s), w_i(\llbracket C \rrbracket(s)), s)$.

End answer..

4. Let $(d_i)_{i\geq 0}$ and $(e_i)_{i\geq 0}$ be chains in a domain $(D, \sqsubseteq)$. Suppose that $(d_i)_{i\geq 0}$ is *majorized by* $(e_i)_{i\geq 0}$, that is: $\forall\, i\, \exists\, j (d_i \sqsubseteq e_j)$.
Prove that $\sqcup_{i\geq 0} d_i \sqsubseteq \sqcup_{i\geq 0} e_i$.
Answer: ....
We are done if we show that $\sqcup_{i\geq 0} e_i$ is an upperbound of the chain $(d_i)_{i\geq 0}$ (because $\sqcup_{i\geq 0} d_i$ is the least upperbound of this chain). This is the case because for every i there is a j such that $d_i \sqsubseteq e_j \sqsubseteq \sqcup_{i\geq 0} e_i$, so for every i, $d_i \sqsubseteq \sqcup_{i\geq 0} e_i$.
End answer..

5. Let $(E, \sqsubseteq')$ be a domain and suppose that in the domain $(D, \sqsubseteq)$, all chains are *eventually constant*, that is: for all chains $(d_i)_{i\geq 0}$ there exists an n such that $d_n = d_{n+1} = d_{n+2} = \ldots$
Show that every monotonic $f : D \to E$ is continuous.
Answer: ....
Let $(d_i)_{i\geq 0}$ be a chain in D. We need to show that $f(\sqcup_{i\geq 0} d_i) \sqsubseteq \sqcup_{i\geq 0} f(d_i)$. (Remember that the other direction of $\sqsubseteq$ always holds.) There is an n such that $d_n = d_{n+1} = \ldots$ and so $f(\sqcup_{i\geq 0} d_i) = f(d_n) \sqsubseteq \sqcup_{i\geq 0} f(d_i)$.
End answer..

6. [Borrowed from Pitts] Consider the two-element domain $\mathbf{2} := \{\bot, \top\}$. For $(D, \sqsubseteq)$ a domain and $d \in D$, define $g_d : D \to \mathbf{2}$ by

$$g_d(x) := \begin{cases} \bot & \text{if } x \sqsubseteq d \\ \top & \text{if } x \not\sqsubseteq d \end{cases}$$

   (a) List all functions $h : \mathbf{2} \to \mathbf{2}$. Which ones are monotonic and which ones are not? Draw a Hasse diagram depicting the ordering of the monotonic functions from $\mathbf{2}$ to $\mathbf{2}$.
   Answer: ....

We have 4 functions from **2** to **2**: Id, $h_\bot$, $h_\top$ and h_{swap} with the following definitions. $\text{Id}(x) = x$, $h_\bot(x) = \bot$, $h_\top(x) = \top$ and $h_{\text{swap}}(\bot) = \top$ and $h_{\text{swap}}(\top) = \bot$. All except h_{swap} are monotonic.
The Hasse diagram has $h_\bot$ as bottom-element and Id and $h_\top$ above that (so it is equivalent to the diagram for $\mathbb{B}_\bot$)
End answer..

(b) Prove that g_d is continuous (for any $d \in D$).
Answer: ....
g_d is monotonic: Assume $x \sqsubseteq y$. We only have to consider the case when $g_d(x) \neq \bot$, so when $g_d(x) = \top$. Then $x \not\sqsubseteq d$. If $y \sqsubseteq d$, then $x \sqsubseteq y \sqsubseteq d$, contradiction, so $y \not\sqsubseteq d$ and $g_d(y) = \top$. Done.
Now Let $(d_i)_{i \in IN}$ be a chain in D. We have to show that $g_d(\sqcup_{i\in\mathbb{N}} d_i) \sqsubseteq \sqcup_{i\in\mathbb{N}} g_d(d_i)$. We only have to consider the case when $\sqcup_{i\in\mathbb{N}} g_d(d_i) = \bot$. then $g_d(d_i) = \bot$ for all $i \in \mathbb{N}$, so $d_i \sqsubseteq d$ for all $i \in \mathbb{N}$. But then $\sqcup_{i\in\mathbb{N}} d_i \sqsubseteq d$, so $g_d(\sqcup_{i\in\mathbb{N}} d_i) = \bot$. Done.
End answer..

(c) Let E be some domain. Prove that $f : E \to D$ is continuous iff $g_d \circ f$ is continuous for all $d \in D$. (You may assume that the composition of two continuous functions is continuous.)
Answer: ....
If $f : E \to D$ is continuous then $g_d \circ f$ is continuous (for each $d \in D$), because composition of continuous functions is continuous. The other way around, assume that $g_d \circ f$ is continuous for each $d \in D$. We first show monotonicity of f: let $x, y \in E$ with $x \sqsubseteq y$. Take $d := f(y)$. Now, $g_d(f(y)) = \bot$ and $g_d \circ f$ is monotonic, so $g_d(f(x)) = \bot$, so $f(x) \sqsubseteq f(y)$.
Now we show that f preserves lubs. Let $(e_i)_{i\in IN}$ be a chain in E. Take $d := \sqcup_{i\in\mathbb{N}} f(e_i)$. Now,

$$\begin{aligned} g_d(f(\sqcup_{i\in\mathbb{N}} e_i)) &= \sqcup_{i\in\mathbb{N}} g_d(f(e_i)) \text{ by continuity of } g_d \circ f \\ &= \bot \text{ because } f(e_i) \sqsubseteq d \quad (\forall\, i \in \mathbb{N}). \end{aligned}$$

So $f(\sqcup_{i\in\mathbb{N}} e_i) \sqsubseteq e = \sqcup_{i\in\mathbb{N}} f(e_i)$. Done.
End answer..