Semantics and Domain theory

Exercises 4

1. Let \((D, \sqsubseteq)\) be the domain of finite and infinite sequences over \(\Sigma := \{a, b\}\) with \(\sqsubseteq\) the prefix ordering. (So \(D = \Sigma^* \cup \Sigma^\omega\).)

(a) Which of the following functions \(f : D \to D\) is monotonic / continuous?

i. \(f(s) = s\) with all \(a\)'s removed.

ii. \(f(s) = abba\) if \(s\) is finite; \(f(s) = s\) if \(s\) is infinite.

iii. \(f(s) = ab\).

iv. \(f(s) = a\) if \(s\) contains finitely many \(b\)'s; \(f(s) = b\) if \(s\) contains infinitely many \(b\)'s

(b) For each of the functions \(f\) in (a) that is continuous, compute the least fixed point of \(f\).

2. Let \((D, \sqsubseteq)\) be a domain with some element \(d_0\) and let \(f : D \to D\) be continuous. Suppose \(d_0 \sqsubseteq f(d_0)\). Prove that \(\sqcup_{i \in \mathbb{N}} f^i(d_0)\) is a fixed point of \(f\).

3. Let \(f, g : (D, \sqsubseteq) \to (D, \sqsubseteq)\) be continuous functions on domain \((D, \sqsubseteq)\). Prove \(\text{fix}(f \circ g) = f(\text{fix}(g \circ f))\)

(a) by unfolding the definition of fix (slide 29)

(b) by using the properties of pre-fixed point (slide 20) and fixed point (slide 29) and proving

i. \(\text{fix}(f \circ g) \sqsubseteq f(\text{fix}(g \circ f))\)

ii. \(f(\text{fix}(g \circ f)) \sqsubseteq \text{fix}(f \circ g)\)

4. For the disjoint union of two domains (also called the binary sum of domains), there are two choices: the coalesced sum (or smashed sum) \(D +c E\), or the separated sum \(D +s E\).

For the coalesced sum, the set \(D +c E\) is defined as

\[
\{\bot\} \cup \{(0, d) \mid d \in D, d \neq \bot_D\} \cup \{(1, e) \mid e \in E, e \neq \bot_E\}
\]

For the separated sum, the set \(D +s E\) is defined as

\[
\{\bot\} \cup \{(0, d) \mid d \in D\} \cup \{(1, e) \mid e \in E\}
\]

So, the separated sum introduces a new \(\bot\) element, whereas the coalesced sum “coalesces (or smashes) them together”.

(NB. The 0 and 1 in the pairs have no special significance, apart from being able to distinguish the “elements coming from \(D\)” from the “elements coming from \(E\)”; we want to define the disjoint union, which should also work, for example, for \(\mathbb{N}_L + \mathbb{N}_L\).)

Let two domains \((D, \sqsubseteq_D)\) and \((E, \sqsubseteq_E)\) be given.

(a) Define the partial ordering \(\sqsubseteq\) on \(D +s E\) and give the \(\bot\)-element.

(b) Define the partial ordering \(\sqsubseteq\) on \(D +c E\) and give the \(\bot\)-element.

(c) For \((f_i)_{i \in \mathbb{N}}\) a chain in \(D +s E\) define \(\sqcup_{i \in \mathbb{N}} f_i\) and prove that it is the least upperbound.

(d) For \((f_i)_{i \in \mathbb{N}}\) a chain in \(D +c E\) define \(\sqcup_{i \in \mathbb{N}} f_i\) and prove that it is the least upperbound.
(e) Define injections \( \text{inl} : D \to D +_s E \) and \( \text{inr} : E \to D +_s E \) that are continuous. (You don’t have to prove that they are continuous.)

(f) Define injections \( \text{inl} : D \to D +_c E \) and \( \text{inr} : E \to D +_c E \) that are continuous. (You don’t have to prove that they are continuous.)

(g) (*) For \( F \) a domain and \( f : D \to F, \ g : E \to F \) we want to define a continuous function \( [f, g] : D + E \to F \) such that \( [f, g](\text{inl}(x)) = f(x) \) and \( [f, g](\text{inr}(x)) = g(x) \).

Show how to define \( [f, g] \) for the case of \( D +_c E \) and for the case of \( D +_s E \). For one of these cases, we can only define \( [f, g] \) if we place additional requirements on \( f \) and \( g \). Which?