Semantics and Domain theory

Exercises 6

1. (a) Show that if $S$ and $T$ are chain-closed, then $S \cup T$ is chain-closed.
   
   (b) Show that if $S_i$ is chain-closed for every $i \in I$, then $\bigcap_{i \in I} S_i$ is chain-closed.

2. (Exercise 4.4.2. of Pitts’ notes) Give an example of a subset $S$ of $D \times D$ that is not chain-closed, but which satisfies:

   (a) $\forall d \in D, \{ e | (d, e) \in S \}$ is chain-closed
   
   (b) $\forall e \in D, \{ d | (d, e) \in S \}$ is chain-closed.

   [Hint: consider $D = D = \Omega$, the cpo in Figure 1.] (Compare this with the property of continuous functions given on Slide 33 of Pitts’ notes.)

3. The collection of chain-closed sets is not closed under arbitrary union. (It is not the case, in general, that $\forall i \in I (S_i$ is chain closed) implies $\bigcup_{i \in I} S_i$ is chain-closed.)

   (a) Conclude this from the previous exercise.
   
   (b) Conclude this by directly constructing a counterexample in $\Omega$.

4. Prove that for $f : D \rightarrow E$ monotonic,

   \[ f^{-1} \text{ preserves chain-closed sets } \Rightarrow f \text{ is continuous}, \]

   where $f^{-1}$ preserves chain-closed sets means that, for all $S \subseteq E$, if $S$ is chain-closed, then $f^{-1}(S)$ is a chain-closed subset of $D$.

5. Show that the untyped $\lambda$-term $\omega (= \lambda x.x \ x)$ is not typable in PCF. That is: show that there are no $\tau_1$ and $\tau_2$ such that $\vdash \text{fn } x : \tau_1, x \ x : \tau_2$.

6. (a) Suppose that the term $\text{mult} : \text{n}at \rightarrow \text{n}at \rightarrow \text{n}at$ defines multiplication in PCF. Give a PCF term that defines the exponentiation function $\text{exp} : \text{n}at \rightarrow \text{n}at \rightarrow \text{n}at$. (So $\text{exp } n \ m$ should denote $n^m$; you don’t have to prove that $\text{exp}$ correctly defines exponentiation.)

   (b) Let $p : \text{n}at \rightarrow \text{n}at$. Define a term $N : \text{n}at$ that denotes the smallest number $n$ such that $p(n) = 0$ and $\forall i < n(p(n) > 0)$. (You don’t have to prove the correctness of $N$.)