Semantics and Domain theory

Exercises 6

1. (a) Show that if S and T are chain-closed, then $S \cup T$ is chain-closed.

   Answer: ....

   Let $(d_i)_{i\in\mathbb{N}}$ be chain in $S \cup T$. Then there are infinitely many i for which $d_i \in S$ or there are infinitely many i for which $d_i \in T$. (It is possible that both are the case, but it is impossible that the chain contains only finitely many elements from S and only finitely many elements from T.)

   Suppose there are infinitely many i for which $d_i \in S$. (The case for T is similar, so we only treat the case for S.) We consider this *sub-chain* of $(d_i)_{i\in\mathbb{N}}$, consisting of the elements that are in S, and we call it $(e_j)_{j\in\mathbb{N}}$. Now, $\sqcup_{i\in\mathbb{N}} d_i = \sqcup_{j\in\mathbb{N}} e_j$ because of standard chain-properties. (For each i there is a j with $d_i \sqsubseteq e_j$ and the other way around: for each j there is a i with $e_j \sqsubseteq d_i$.) Now, $\sqcup_{j\in\mathbb{N}} e_j \in S$, beacuse S is chain closed and so $\sqcup_{i\in\mathbb{N}} d_i \in S \cup T$.

   End answer ....

   (b) Show that if S_i is chain-closed for every $i \in I$, then $\bigcap_{i\in I} S_i$ is chain-closed.

   Answer: ....

   Let $(d_j)_{j\in\mathbb{N}}$ be chain in $\bigcap_{i\in I} S_i$. Then $(d_j)_{j\in\mathbb{N}}$ is a chain in each of the S_i (for $i \in I$). As each S_i is chain-closed, $\sqcup_{j\in\mathbb{N}} d_j \in S_i$ for each $i \in I$, so $\sqcup_{j\in\mathbb{N}} d_j \in \bigcap_{i\in I} S_i$.

   End answer ....

2. (Exercise 4.4.2. of Pitts' notes) Give an example of a subset S of $D \times D$ that is not chain-closed, but which satisfies:

$$\begin{array}{ll} (a) & \forall d \in D,\ \{e | (d, e) \in S\} \text{ is chain-closed} \\ (b) & \forall e \in D,\ \{d | (d, e) \in S\} \text{ is chain-closed.} \end{array}$$

   [Hint: consider $D = D = \Omega$, the cpo in Figure 1.] (Compare this with the property of continuous functions given on Slide 33 of Pitts' notes.)

   Answer: ....

   Take $D := \Omega$, the well-known domain with elements $\mathbb{N} \cup \{\omega\}$. Define

$$S := \{(n, n) | n \in \mathbb{N}\}.$$

   (Note: this looks very much like the set Δ that we have defined generally for each domain D, but note that we have removed (ω, ω) from Δ.)

   Now, for a fixed $d \in \Omega$, $\{e \in \Omega | (d, e) \in S\}$ is chain-closed, because if $d \in \mathbb{N}$, then $\{e \in \Omega | (d, e) \in S\} = \{d\}$ and if $d = \omega$, then $\{e \in \Omega | (d, e) \in S\} = \emptyset$. Similarly, for a fixed $e \in \Omega$, $\{d \in \Omega | (d, e) \in S\}$ is chain-closed.

   On the other hand, S is not chain-closed: the lub of the chain $(i, i)_{i\in\mathbb{N}}$ is (ω, ω) which is not in S.

   End answer ....

3. The collection of chain-closed sets is not closed under arbitrary union. (It is not the case, in general, that $\forall i \in I(S_i$ is chain closed) implies $\bigcup_{i\in I} S_i$ is chain-closed.)

   (a) Conclude this from the previous exercise.
   Answer: ....
   See the answer of the previous exercise. For each $i \in \mathbb{N}$, $S_i := \{e \in \Omega | (i, e) \in S\}$ is chain-closed, but $S = \cup_{i\in\mathbb{N}} S_i$ is not chain closed.
   End answer...

   (b) Conclude this by directly constructing a counterexample in Ω.

   Answer: ....
   $S_i := \{i\}$ is a chain-closed subset of Ω for each $i \in \mathbb{N}$, but $\cup_{i\in\mathbb{N}}\{i\} = \mathbb{N}$ is not chain-closed.
   End answer...

4. Prove that for $f : D \to E$ monotonic,

$$f^{-1} \text{ preserves chain-closed sets} \Rightarrow f \text{ is continuous},$$

   where f^{-1} *preserves chain-closed sets* means that, for all $S \subseteq E$, if S is chain-closed, then $f^{-1}(S)$ is a chain-closed subset of D.

   Answer: ....
   Suppose that f is monotonic and f^{-1} preserves chain-closed sets. Let $(d_i)_{i\in\mathbb{N}}$ be a chain in D. We need to prove that $f(\sqcup_{i\in I} d_i) \sqsubseteq \sqcup_{i\in\mathbb{N}} f(d_i)$. We consider the following set $S \subseteq E$: $S := \{f(d_i) | i \in I\} \cup \{\sqcup_{i\in\mathbb{N}} f(d_i)\}$. This set is clearly chain closed. So $f^{-1}(S)$ is chain-closed. Let's take a look at $f^{-1}(S)$: it contains all d_i (because $f(d_i) \in S$ for all $i \in \mathbb{N}$). Therefore $f^{-1}(S)$ contains $\sqcup_{i\in\mathbb{N}} d_i$. This means that $f(\sqcup_{i\in\mathbb{N}} d_i) \in S$, which implies that $f(\sqcup_{i\in\mathbb{N}} d_i) \sqsubseteq \sqcup_{i\in\mathbb{N}} f(d_i)$.

   End answer...

5. Show that the untyped λ-term ω $(= \lambda x.x\,x)$ is not typable in PCF. That is: show that there are no τ_1 and τ_2 such that $\vdash \mathbf{fn}\ x : \tau_1.\,(x\ x) \ : \ \tau_2$.

   Answer: ....
   Suppose $\vdash \mathbf{fn}\ x : \tau_1.\,(x\ x) \ : \ \tau_2$ for some types τ_1 and τ_2. We consider the possible derivation of this judgment and show that such a derivation cannot exist.

   A derivation looks like this:

$$\dfrac{\dfrac{x : \tau_1 \vdash x \ : \ \sigma_1 \to \sigma_2 \qquad x : \tau_1 \vdash x \ : \ \sigma_1}{x : \tau_1 \vdash x\ x \ : \ \tau_2}\,(\text{:app})}{\vdash \mathbf{fn}\ x : \tau_1.\,(x\ x) \ : \ \tau_2}\,(\text{:fn})$$

   where we must have $\tau_1 = \sigma_1 \to \sigma_2$ and $\tau_1 = \sigma_1$ and $\tau_2 = \sigma_2$. Then $\sigma_1 \to \sigma_2 = \sigma_1$, but there are no types in PCF for which this equation holds. So, there is no such typing derivation.

   End answer...

6. (a) Suppose that the term mult : **nat** $\to$ **nat** $\to$ **nat** defines multiplication in PCF. Give a PCF term that defines the exponentiation function exp : **nat** $\to$ **nat** $\to$ **nat**. (So $\exp n\, m$ should denote n^m; you don't have to prove that exp correctly defines exponentiation.)

Answer: ....

We first write down the fixed-point equation that exp should satisfy: for n, m : **nat** we want

$$\exp n\, m = \textbf{if}\,\textbf{zero}(m)\,\textbf{then}\,\underline{1}\,\textbf{else}\,(\text{mult}\, n\, (\exp n\, (\textbf{pred}\, m))$$

So we take exp := **fix** f where f : (**nat**→**nat**→**nat**) → (**nat**→**nat**→**nat**) is defined by

$$\begin{aligned} f := \quad & \textbf{fn}\, e : \textbf{nat}{\to}\textbf{nat}{\to}\textbf{nat}.\textbf{fn}\, n, m : \textbf{nat}. \\ & \textbf{if}\,\textbf{zero}(m)\,\textbf{then}\,\underline{1}\,\textbf{else}\,(\text{mult}\, n\, (e\, n\, (\textbf{pred}\, m)) \end{aligned}$$

End answer..

(b) Let p : **nat** $\to$ **nat**. Define a term N : **nat** that denotes the smallest number n such that $p(n) = 0$ and $\forall i < n(p(n) > 0)$. (You don't have to prove the correctness of N.)

Answer: ....

We first write down the fixed-point equation that N should satisfy. We cannot do this directly so we introduce a "helper function" search : **nat** $\to$ **nat**, which searches N starting from a start-value n:

$$\text{search}\, n = \textbf{if}\,\textbf{zero}(p(n))\,\textbf{then}\, n\,\textbf{else}\,(\text{search}\,(\textbf{succ}\, n))$$

We define search := **fix** f where f : (**nat**→**nat**) → (**nat**→**nat**) is defined by

$$f := \textbf{fn}\, s : \textbf{nat}{\to}\textbf{nat}.\textbf{fn}\, n : \textbf{nat}.\textbf{if}\,\textbf{zero}(p(n))\,\textbf{then}\, n\,\textbf{else}\,(s\,(\textbf{succ}\, n))$$

Now we take N := search **zero**.

End answer..