Semantics and Domain theory

Exercises 7

1. Show that, if $Q \Downarrow \tau$ V, then $(\text{fn}\ x : \tau. \text{fn}\ y : \tau.y)PQ \Downarrow V$. (NB. $V$ denotes an arbitrary value.)

**Answer:** Suppose we have a derivation of $Q \Downarrow \tau$ V. Then we have the following derivation of $(\text{fn}\ x : \tau. \text{fn}\ y : \tau.y)PQ \Downarrow V$, where we write $F$ for $\text{fn}\ x : \tau. \text{fn}\ y : \tau.y$.

\[
F \Downarrow \text{fn}\ x : \tau. \text{fn}\ y : \tau.y \quad (\text{fn}\ y : \tau.y)[P/x] \Downarrow \text{fn}\ y : \tau.y \quad \Downarrow\text{app} \\
FP \Downarrow \text{fn}\ y : \tau.y \quad \Downarrow\text{app} \\
FPQ \Downarrow V
\]

End answer

2. To prove that PCF evaluation is deterministic, we prove (in Proposition 5.4.1) that the following set is closed under the rules of Fig.3

\[\{(M, \tau, V)| M \Downarrow \tau V \land \forall V'(M \Downarrow \tau V' \Rightarrow V = V')\}\]

Show this for the cases of the rules ($\Downarrow\text{if1}$) and ($\Downarrow\text{cbn}$).

(An alternative way of looking at this is to prove the following:

$M \Downarrow \tau V \Rightarrow \forall V'(M \Downarrow \tau V' \Rightarrow V = V')$

by induction on the derivation of $M \Downarrow \tau V$. Do only the cases when the last applied rule is ($\Downarrow\text{if1}$) or ($\Downarrow\text{cbn}$).)

**Answer:** The case for $\Downarrow\text{if1}$:

\[
\begin{align*}
M \Downarrow \text{true} & \quad N \Downarrow V \\
\text{if } M \text{ then } N \text{ else } P \Downarrow V & \Downarrow\text{if1}
\end{align*}
\]

We have the induction hypothesis for $M \Downarrow \text{true}$ and for $N \Downarrow V$.

If we have a derivation of $M \Downarrow \text{true} N \Downarrow V$, then the last applied rules must be $\Downarrow\text{if1}$ or $\Downarrow\text{if2}$. As $M \Downarrow \text{true}$, it follows from (IH) that $M \Downarrow \text{false}$. So the last rule has been $\Downarrow\text{if1}$ and we have

\[
\begin{align*}
M \Downarrow \text{true} & \quad N \Downarrow V' \\
\text{if } M \text{ then } N \text{ else } P \Downarrow V' & \Downarrow\text{if1}
\end{align*}
\]

Now, (IH) for $N \Downarrow V$ says that $V = V'$ and we are done.

The case for $\Downarrow\text{cbn}$:

\[
\begin{align*}
M \Downarrow \text{fn}\ x : \tau. N & \quad N[Q/x] \Downarrow V \\
M Q \Downarrow V & \Downarrow\text{cbn}
\end{align*}
\]
We have the induction hypothesis for $M \Downarrow \text{fn} : \tau$. Now, (IH) for $M N \Downarrow V'$, then the last applied rules must be \$\downarrow_{\text{cbn}}$. So we have

$$
\begin{array}{c}
M \Downarrow y : \tau'. N' \Downarrow Q[y]/y \Downarrow V' \\
\hline
M Q \Downarrow V' \\
\hline
\end{array}
$$

Now, (IH) for $M \Downarrow \text{fn} : \tau. N$ says that $\text{fn} : \tau. N = \text{fn} : \tau. N'$ and so $N[Q/x] = N'[Q/y]$. We are done by the (IH) for $N[Q/x] \Downarrow V$.

**End answer.**

3. Prove that the following terms $M$ and $N$ are not contextually equivalent.

(a) $M = \text{if } x \text{ then } 0 \text{ else } 1$ and $N = \text{if } y \text{ then } 0 \text{ else } 1$.

**Answer:**

Take $C[-] = (\text{fn } x : \text{bool. fn } y : \text{bool. -}) \text{false true.}$ Then $C[M] \Downarrow 1$ and $C[N] \Downarrow 0$. The derivation of the first, writing $F$ for $\text{fn } x : \text{bool. fn } y : \text{bool. if } x \text{ then } 0 \text{ else } 1$ and $G$ for $\text{fn } y : \text{bool. if false then } 0 \text{ else } 1$ and

$$
\begin{array}{c}
F \Downarrow F \\
\hline
\text{false } \Downarrow G \\
\hline
\text{false false } \Downarrow 1 \\
\hline
\text{false true } \Downarrow 1 \\
\hline
\end{array}
$$

$$
\text{false true } \Downarrow 1
$$

Also, one could take $C[-] = (\text{fn } x : \text{bool. fn } y : \text{bool. -}) \Omega_{\text{bool true.}}$. Then $C[M] \not\Downarrow$ and $C[N] \Downarrow 0$.

**End answer.**

(b) $M = \text{fn } x : \text{nat.succ(pred } x) \text{ and } N = \text{fn } x : \text{nat.x}$.

**Answer:**

Take $C[-] = -0$. Then $C[M] \not\Downarrow$ and $C[N] \Downarrow 0$. The derivation of the second is easy, and we don’t give it. For the first, suppose it has a derivation. Then show that this derivation should contain a subderivation of $\text{pred } 0 \Downarrow V$ for some value $V$, but that derivation doesn’t exist.

**End answer.**

4. (a) Give a type $\tau$, a term $M$, values $V$, $V'$ and a context $C[-]$ such that $M \Downarrow \tau$ but $C[M] \Downarrow \tau' \neq C[\tau]$.

**Answer:**

(So $M \Downarrow V \not\Downarrow C[M] \Downarrow C[\tau]$.) A simple example is $M = \text{pred(succ } 0)$, $C[-] = \text{fn } x : \text{nat. -}$, because we have $M \Downarrow 0$ and $C[M] = \text{fn } x : \text{nat. pred (succ } 0) \Downarrow \text{fn } x : \text{nat.pred (succ } 0) \neq \text{fn } x : \text{nat. 0} = C[0]$.

NB. We don’t give the derivations, but in an exam you should give them.

**End answer.**

(b) Give a type $\tau$, a term $M$, a value $V$ and a context $C[-]$ such that $M \Downarrow \tau$ but $C[M] \not\Downarrow \tau$ ($C[M]$ has no value.)

**Answer:**

...
(So \( M \downarrow \tau V \not\Rightarrow \exists V'(C[M] \downarrow \tau V') \).) A very simple example is \( M = 0, C[-] = \text{pred}(-) \), because we have \( M \downarrow 0 \) and \( C[M] = \text{pred}(0) \not\downarrow \). A different example is \( M = 0, C[-] = \text{if zero}(-) \text{ then } \Omega_{\text{nat}} \text{ else } - \), because we have \( M \downarrow 0 \) and \( C[M] = \text{if zero}(0) \text{ then } \Omega_{\text{nat}} \text{ else } 0 \not\downarrow \).

NB. We don’t give the derivations, but in an exam you should give them.

End answer

(c) Give a type \( \tau \), a term \( M \), a value \( V \) and a context \( C[-] \) such that \( M \not\downarrow \tau \) but \( C[M] \downarrow \tau V \)

Answer: 

5. Given the definition of plus (Exercise 5.6.3.)

\[
\begin{align*}
\text{plus} &= \text{fix}(\text{fn} p : \text{nat} \to \text{nat}. \text{fn} x : \text{nat}. \text{fn} y : \text{nat}. \\
&\quad \text{if zero}(y) \text{ then } x \text{ else succ}(p \times \text{pred}(y))
\end{align*}
\]

Prove (by induction) that

\[\forall m, n(\text{plus} \times \text{succ}^m(0) \times \text{succ}^n(0) \downarrow_{\text{nat}} \text{succ}^{m+n}(0))\]

NB. First identify the proper statement that you need and that you can prove relatively easily by induction.

Answer: 

We introduce some abbreviations

\[
\begin{align*}
n &:= \text{succ}^n(0), \text{ for } n \in \mathbb{N} \\
A(p, x, y) &:= \text{if zero}(y) \text{ then } x \text{ else succ}(p \times \text{pred}(y)) \\
B(p) &:= \text{fn} x : \text{nat}. \text{fn} y : \text{nat}. A(p, x, y) \\
H &:= \text{fn} p : \text{nat} \to \text{nat} \to \text{nat}. B(p) \\
T(y) &:= A(\text{plus}, m, y)
\end{align*}
\]

We prove, by induction on \( n \), for all \( Q \),

\[ Q \downarrow n \implies T(Q) \downarrow m + n. \]

So, \( T(Q) \) is if \( \text{zero}(Q) \text{ then } m \text{ else succ}(\text{plus } m \times \text{pred}(Q)) \) and plus is defined as \( \text{fix} H \).

Case \( n = 0 \): immediate by one application of \((\downarrow)\).

Case \( n + 1 \): The IH is \( Q. (Q \downarrow n \implies T(Q) \downarrow m + n) \). Now, suppose that \( Q \downarrow n + 1 \) we need to prove \( T(Q) \downarrow m + n + 1 \).
We use the fact that, if $Q \Downarrow n + 1$, then $\text{pred}(Q) \Downarrow n$.
The derivation is as follows: (You have to fill in the dots yourself and the double line requires two steps.)

\[
\begin{array}{c}
H \Downarrow B(+) \\
\text{IH} \\
\Downarrow B(+) \\
\text{ pred}(Q) \Downarrow m + n \\
\text{ plus } m \text{ pred}(Q) \Downarrow m + n \\
\text{ succ(plus } m \text{ pred}(Q)) \Downarrow m + n + 1 \\
T(Q) \Downarrow m + n + 1
\end{array}
\]

Now, the result, $\text{ plus } m n \Downarrow m + n$, follows immediately:

\[
\begin{array}{c}
H \Downarrow B(+) \\
\text{ IH} \\
\Downarrow B(+) \\
\text{ plus } m n \Downarrow m + n \\
\text{ succ(plus } m n \Downarrow m + n + 1 \\
T(Q) \Downarrow m + n + 1
\end{array}
\]

End answer