Semantics and Domain theory
Exercises 10
Exercises from the dl-domains notes

1. Suppose that a monotonic function \( p : (\mathbb{B}_\perp \times \mathbb{B}_\perp) \to \mathbb{B}_\perp \) satisfies
   
   - \( p(tt, \bot) = tt \),
   - \( p(\bot, tt) = tt \),
   - \( p(ff, ff) = ff \).
   
   Show that \( p \) coincides with the parallel-or function (on Slide 72 of the notes of Pitts) in the sense that \( p(d_1, d_2) = \text{por}(d_1)(d_2) \), for all \( d_1, d_2 \in \mathbb{B}_\perp \).

   **Answer:**
   As \( p \) is monotonic, we have \( p(tt, ff) = p(ff, tt) = p(tt, tt) = tt \). This leaves 3 entries in the table: \( p(\bot, \bot) \), \( p(ff, \bot) \) and \( p(\bot, ff) \). As \( (ff, \bot) \subseteq (ff, ff) \) and \( (ff, \bot) \subseteq (ff, tt) \), we must have \( p(ff, \bot) \subseteq p(ff, ff) \) and \( p(ff, \bot) \subseteq p(ff, tt) \), and therefore \( p(ff, \bot) = \bot \). Similarly \( p(\bot, ff) = \bot \), and therefore \( p(\bot, \bot) = \bot \).

   **End answer**

2. Show that the evaluation relation for PCF+por (Slide 77, where rules for \( \text{por}(M_1, M_2) \Downarrow V \) have been added to PCF) is still deterministic: If \( M \Downarrow V \), then for all \( V' \), if \( M \Downarrow V' \), then \( V = V' \). This is again proved by induction on the derivation of \( M \Downarrow V \); do the case for the new por-rules.

   **Answer:**
   We have to prove that, if \( M \Downarrow V \), then for all \( V' \), if \( M \Downarrow V' \), then \( V = V' \). This is proved by induction on the derivation of \( M \Downarrow V \). We do the case for the last rule being the por-rules:

   
   \[
   \frac{M_1 \Downarrow \text{true}}{\text{por}(M_1, M_2) \Downarrow \text{true}} \quad \frac{M_2 \Downarrow \text{true}}{\text{por}(M_1, M_2) \Downarrow \text{true}} \quad \frac{M_1 \Downarrow \text{false} \quad M_2 \Downarrow \text{false}}{\text{por}(M_1, M_2) \Downarrow \text{false}} \quad \frac{M_1 \Downarrow \text{false}}{\text{por}(M_1, M_2) \Downarrow \text{false}}
   \]

   We only treat \( \text{por}_1 \) and \( \text{por}_3 \). (The rule \( \text{por}_2 \) is similar to \( \text{por}_1 \).) Suppose

   \[
   \frac{M_1 \Downarrow \text{true}}{\text{por}(M_1, M_2) \Downarrow \text{true}}
   \]

   Now, if \( \text{por}(M_1, M_2) \Downarrow V' \), this could have been obtained by \( \text{por}_1 \), \( \text{por}_2 \) or \( \text{por}_3 \).

   - (\( \text{por}_1 \)) Then \( V' = \text{true} \) and done.
   - (\( \text{por}_2 \)) Then also \( V' = \text{true} \) and done.
   - (\( \text{por}_3 \)) Then \( M_1 \Downarrow \text{false} \) and by induction hypothesis: \( \text{false} = \text{true} \), contradiction, so (\( \text{por}_3 \)) is not the last rule.

   Suppose

   \[
   \frac{M_1 \Downarrow \text{false} \quad M_2 \Downarrow \text{false}}{\text{por}(M_1, M_2) \Downarrow \text{false}}
   \]

   Now, if \( \text{por}(M_1, M_2) \Downarrow V' \), this could have been obtained by \( \text{por}_1 \), \( \text{por}_2 \) or \( \text{por}_3 \).

   - (\( \text{por}_1 \)) Then \( V' = \text{true} \) and by induction hypothesis: \( \text{false} = \text{true} \), contradiction, so (\( \text{por}_1 \)) is not the last rule.
   - (\( \text{por}_2 \)) Then \( V' = \text{true} \) and by induction hypothesis: \( \text{false} = \text{true} \), contradiction, so (\( \text{por}_2 \)) is not the last rule.
(\text{\texttt{por3}}) Then \( V' = \text{false} \) and done.

\textbf{End answer} .................................................................

3. (a) Describe the compact elements of \( \mathbb{N}_\perp \to \mathbb{N}_\perp \) and prove that these are indeed the compact elements.

\textbf{Answer:} .................................................................

We look at the domain, \( \text{dom}(f) \) of a function \( f : \mathbb{N}_\perp \to \mathbb{N}_\perp \), which is defined as \( \text{dom}(f) = \{ x \in \mathbb{N}_\perp \mid f(x) \neq \bot \} \). Remember that \( f \sqsubseteq g \) iff \( \text{dom}(f) \sqsubseteq \text{dom}(g) \land \forall x \in \text{dom}(f) \ (f(x) = g(x)) \).

If \( f : \mathbb{N}_\perp \to \mathbb{N}_\perp \) has an \textbf{infinite domain}, it is not compact: the set

\[ F := \{ f_Y \mid \text{dom}(f_Y) = Y \subseteq \text{dom}(f), Y \text{ is finite and } \forall x \in Y (f_Y(x) = f(x)) \} \]

is a directed set and \( \bigsqcup F = f \), but there is no \( f_Y \) such that \( f \sqsubseteq f_Y \). (NB. That \( F \) is directed can be seen as follows: if \( f_Y \) and \( f_Z \) are elements of \( F \), then \( f_Y \sqcup f_Z \) is also an element of \( F \) and \( f_Y, f_Z \sqsubseteq f_{Y \cup Z} \).)

If \( f : \mathbb{N}_\perp \to \mathbb{N}_\perp \) has a \textbf{finite domain}, it is compact: let \( \{ x_1, \ldots, x_n \} \) be the domain of \( f \) and let \( F \) be a directed set of functions with \( f \sqsubseteq \bigsqcup F \). Then for every \( x_i \) there is a \( g_i \in F \) with \( x_i \in \text{dom}(g_i) \) and \( g_i(x_i) = f(x_i) \). As \( F \) is directed, there is a \( g \in F \) with \( g_1, \ldots, g_n \sqsubseteq g \). But then \( \text{dom}(f) \sqsubseteq \text{dom}(g) \) and so \( f \sqsubseteq g \).

So, the \textbf{basis} of the domain \( \mathbb{N}_\perp \to \mathbb{N}_\perp \) is

\[ \mathcal{B}_{\mathbb{N}_\perp \to \mathbb{N}_\perp} = \{ f \mid f : \mathbb{N}_\perp \to \mathbb{N}_\perp \text{ with } \text{dom}(f) \text{ finite} \} \]

\textbf{End answer} .................................................................

(b) Show that \( \mathbb{N}_\perp \to \mathbb{N}_\perp \) is an algebraic dcpo.

\textbf{Answer:} .................................................................

Given \( f : \mathbb{N}_\perp \to \mathbb{N}_\perp \), consider:

\[ F := \{ g \mid g \in \mathcal{B}_{\mathbb{N}_\perp \to \mathbb{N}_\perp} \text{ and } g \sqsubseteq f \}. \]

Then the \( g \) in \( F \) are exactly the functions with a finite domain such that \( g \sqsubseteq f \).

For every \( x \in \text{dom}(f) \) there is a \( g \in F \) with \( x \in \text{dom}(g) \) and \( g(x) = f(x) \). So \( f \sqsubseteq \bigsqcup F \).

\textbf{End answer} .................................................................

4. Let \( X \) be a set and let \( \wp(X) \) be the power set of \( X \) ordered by inclusion \( \subseteq \).

(a) Describe the compact elements of \( \wp(X) \) and prove that these are indeed the compact elements.

\textbf{Answer:} .................................................................

If \( Y \subseteq X \) and \( Y \) is \textbf{infinite}, it is not compact: the set

\[ Y_{\text{fin}} := \{ Z \mid Z \subseteq Y, Z \text{ finite} \} \]

is a directed set and \( \bigsqcup Y_{\text{fin}} \not\sqsubseteq Y \), but there is no \( Z \in Y_{\text{fin}} \) such that \( Y \sqsubseteq Z \).

NB1. \( Y_{\text{fin}} \) is directed because if \( Z_1 \) and \( Z_2 \) are elements of \( Y_{\text{fin}} \), then
$Z_1 \cup Z_2$ is also an element of $Y$ and $Z_1, Z_2 \subseteq Z_1 \cup Z_2$.

NB2. For $\bigcup Y_{\text{fin}} \equiv Y$: If $y \in \bigcup Y_{\text{fin}}$, then $y \in Z \subseteq Y$ for some $Z$, so $y \in Y$. If $y \in Y$, then $y \in \{y\}$ and $\{y\} \in Y_{\text{fin}}$, so $y \in \bigcup Y_{\text{fin}}$.

If $Y \subseteq X$ and $Y$ is finite, it is compact: suppose $Y = \{y_1, \ldots, y_n\}$ and let $W$ be a directed set of subsets with $\bigcup W = Y$. Then for every $y_i$ there is a $Z_i \in W$ with $y_i \in Z_i$. As $W$ is directed, there is a $Z \in W$ with $Z_1 \cup \ldots \cup Z_n \subseteq Z$. But then $Y \subseteq Z$.

So, the basis of the domain $\wp(X)$ is

$$B_{\wp(X)} = \{Y \mid Y \subseteq X \text{ with } Y \text{ finite}\}$$

and note that if $X$ is finite, then $B_{\wp(X)} = \wp(X)$ (all elements are compact).

End answer

(b) Show that $\wp(X)$ is an algebraic dcpo.

**Answer:**

Given $Y \subseteq X$, consider $Y_{\text{fin}} := \{Z \mid Z \subseteq Y, Z \in B_{\wp(X)}\}$. Then

$$Y_{\text{fin}} = \{Z \mid Z \subseteq Y, Z \text{ finite}\}$$

and we need to show that $Y = \bigcup Y_{\text{fin}}$, which we have done in the answer to Exercise 4a: see the NB2.

End answer

5. Suppose we are in a dcpo where each pair of elements has a glb. Show that $\forall x, y (x \subseteq y \iff x = x \sqcap y)$.

6. (a) Prove that $\mathbb{N}_\bot$ satisfies (axiom d):

$$\forall x, y, z \in \mathbb{N}_\bot (y \uparrow z \Rightarrow x \sqcap (y \sqcup z) = (x \sqcap y) \sqcup (x \sqcap z)).$$

(b) Prove that $\mathbb{N}_\bot \rightarrow \mathbb{N}_\bot$ (the set of Scott continuous functions) satisfies (axiom d).

7. We are in a bounded complete p.o. and we consider the property $(\ast)$ (used in stability)

$$\forall x, y \in D (x \uparrow y \rightarrow f(x \sqcap y) = f(x) \sqcap f(y)) \quad (\ast)$$

Show that, if $f$ satisfies $(\ast)$, then it is monotone.

8. (a) Define all possible different “AND” functions as monotone functions (in $\mathbb{B}_\bot \times \mathbb{B}_\bot \rightarrow \mathbb{B}_\bot$).

(b) Show that 3 of your functions can be defined in PCF.

(c) Now show that one of your functions cannot be defined in PCF

i. by semantic means (using dl-domains).

ii. by using the non-definability of $\text{por}$

9. Prove that the identity function $I : \mathbb{N}_\bot \rightarrow \mathbb{N}_\bot$ is not very finite.

10. Prove that, in a bounded complete dcpo, every non-empty set $X$ has a greatest lower bound, $\bigsqcap X$. 