Semantics and Domain theory

Exercises 10

Exercises from the dI-domains notes

1. Suppose that a monotonic function $p : (\mathbb{B}_\bot \times \mathbb{B}_\bot) \to \mathbb{B}_\bot$ satisfies

   - $p(\mathrm{tt}, \bot) = \mathrm{tt}$,
   - $p(\bot, \mathrm{tt}) = \mathrm{tt}$,
   - $p(\mathrm{ff}, \mathrm{ff}) = \mathrm{ff}$.

   Show that p coincides with the parallel-or function (on Slide 72 of the notes of Pitts) in the sense that $p(d_1, d_2) = \mathrm{por}(d_1)(d_2)$, for all $d_1, d_2 \in \mathbb{B}_\bot$.

   Answer: ....

   As p is monotonic, we have $p(\mathrm{tt}, \mathrm{ff}) = p(\mathrm{ff}, \mathrm{tt}) = p(\mathrm{tt}, \mathrm{tt}) = \mathrm{tt}$. This leaves 3 entries in the table: $p(\bot, \bot)$, $p(\mathrm{ff}, \bot)$ and $p(\bot, \mathrm{ff})$. As $(\mathrm{ff}, \bot) \sqsubseteq (\mathrm{ff}, \mathrm{ff})$ and $(\mathrm{ff}, \bot) \sqsubseteq (\mathrm{ff}, \mathrm{tt})$, we must have $p(\mathrm{ff}, \bot) \sqsubseteq p(\mathrm{ff}, \mathrm{ff})$ and $p(\mathrm{ff}, \bot) \sqsubseteq p(\mathrm{ff}, \mathrm{tt})$, and therefore $p(\mathrm{ff}, \bot) = \bot$. Similarly $p(\bot, \mathrm{ff}) = \bot$, and therefore $p(\bot, \bot) = \bot$.

   End answer..

2. Show that the evaluation relation for PCF+por (Slide 77, where rules for $\mathbf{por}(M_1, M_2) \Downarrow V$ have been added to PCF) is still deterministic: If $M \Downarrow V$, then for all V', if $M \Downarrow V'$, then $V = V'$. This is again proved by induction on the derivation of $M \Downarrow V$; do the case for the new **por**-rules.

   Answer: ....

   We have to prove that, if $M \Downarrow V$, then for all V', if $M \Downarrow V'$, then $V = V'$. This is proved by induction on the derivation of $M \Downarrow V$. We do the case for the last rule being the **por**-rules:

$$\frac{M_1 \Downarrow \mathbf{true}}{\mathbf{por}(M_1, M_2) \Downarrow \mathbf{true}} \Downarrow_{\mathbf{por1}} \qquad \frac{M_2 \Downarrow \mathbf{true}}{\mathbf{por}(M_1, M_2) \Downarrow \mathbf{true}} \Downarrow_{\mathbf{por2}} \qquad \frac{M_1 \Downarrow \mathbf{false} \quad M_2 \Downarrow \mathbf{false}}{\mathbf{por}(M_1, M_2) \Downarrow \mathbf{false}} \Downarrow_{\mathbf{por3}}$$

We only treat **por**1 and **por**3. (The rule **por**2 is similar to **por**1.) Suppose

$$\frac{M_1 \Downarrow \mathbf{true}}{\mathbf{por}(M_1, M_2) \Downarrow \mathbf{true}} \Downarrow_{\mathbf{por1}}$$

Now, if $\mathbf{por}(M_1, M_2) \Downarrow V'$, this could have been obtained by $\Downarrow_{\mathbf{por1}}$, $\Downarrow_{\mathbf{por2}}$ or $\Downarrow_{\mathbf{por3}}$.

($\Downarrow_{\mathbf{por1}}$) Then $V' = \mathbf{true}$ and done.

($\Downarrow_{\mathbf{por2}}$) Then also $V' = \mathbf{true}$ and done.

($\Downarrow_{\mathbf{por3}}$) Then $M_1 \Downarrow \mathbf{false}$ and by induction hypohesis: $\mathbf{false} = \mathbf{true}$, contradiction, so ($\Downarrow_{\mathbf{por3}}$) is not the last rule.

Suppose

$$\frac{M_1 \Downarrow \mathbf{false} \quad M_2 \Downarrow \mathbf{false}}{\mathbf{por}(M_1, M_2) \Downarrow \mathbf{false}} \Downarrow_{\mathbf{por3}}$$

Now, if $\mathbf{por}(M_1, M_2) \Downarrow V'$, this could have been obtained by $\Downarrow_{\mathbf{por1}}$, $\Downarrow_{\mathbf{por2}}$ or $\Downarrow_{\mathbf{por3}}$.

($\Downarrow_{\mathbf{por1}}$) Then $V' = \mathbf{true}$ and by induction hypohesis: $\mathbf{false} = \mathbf{true}$, contradiction, so ($\Downarrow_{\mathbf{por1}}$) is not the last rule.

($\Downarrow_{\mathbf{por2}}$) Then $V' = \mathbf{true}$ and by induction hypohesis: $\mathbf{false} = \mathbf{true}$, contradiction, so ($\Downarrow_{\mathbf{por2}}$) is not the last rule.

($\Downarrow_{\mathbf{por3}}$) Then $V' = \mathbf{false}$ and done.

End answer . . . .

3. (a) Describe the compact elements of $\mathbb{N}_\perp \to \mathbb{N}_\perp$ and prove that these are indeed the compact elements.

Answer: . . . .
We look at the domain, $\mathrm{dom}(f)$ of a function $f : \mathbb{N}_\perp \to \mathbb{N}_\perp$, which is defined as $\mathrm{dom}(f) = \{x \in \mathbb{N}_\perp \mid f(x) \neq \perp\}$. Remember that $f \sqsubseteq g$ iff $\mathrm{dom}(f) \subseteq \mathrm{dom}(g) \wedge \forall x \in \mathrm{dom}(f)\,(f(x) = g(x))$.
If $f : \mathbb{N}_\perp \to \mathbb{N}_\perp$ has an **infinite domain**, it is not compact: the set

$$F := \{f_Y \mid \mathrm{dom}(f_Y) = Y \subseteq \mathrm{dom}(f), Y \text{ is finite and } \forall x \in Y (f_Y(x) = f(x))\}$$

is a directed set and $\bigsqcup F = f$, but there is no f_Y such that $f \sqsubseteq f_Y$. (NB. That F is directed can be seen as follows: if f_Y and f_Z are elements of F, then $f_{Y \cup Z}$ is also an element of F and $f_Y, f_Z \sqsubseteq f_{Y \cup Z}$.)
If $f : \mathbb{N}_\perp \to \mathbb{N}_\perp$ has a **finite domain**, it is compact: let $\{x_1, \ldots, x_n\}$ be the domain of f and let F be a directed set of functions with $f \sqsubseteq \bigsqcup F$. Then for every x_i there is a $g_i \in F$ with $x_i \in \mathrm{dom}(g_i)$ and $g_i(x_i) = f(x_i)$. As F is directed, there is a $g \in F$ with $g_1, \ldots, g_n \sqsubseteq g$. But then $\mathrm{dom}(f) \subseteq \mathrm{dom}(g)$ and so $f \sqsubseteq g$.
So, the *basis* of the domain $\mathbb{N}_\perp \to \mathbb{N}_\perp$ is

$$\mathbf{B}_{\mathbb{N}_\perp \to \mathbb{N}_\perp} = \{f \mid f : \mathbb{N}_\perp \to \mathbb{N}_\perp \text{ with } \mathrm{dom}(f) \text{ finite}\}$$

End answer . . . .

(b) Show that $\mathbb{N}_\perp \to \mathbb{N}_\perp$ is an algebraic dcpo.
Answer: . . . .
Given $f : \mathbb{N}_\perp \to \mathbb{N}_\perp$, consider:

$$F := \{g \mid g \in \mathbf{B}_{\mathbb{N}_\perp \to \mathbb{N}_\perp} \text{ and } g \sqsubseteq f\}.$$

Then the g in F are exactly the functions with a finite domain such that $g \sqsubseteq f$.
For every $x \in \mathrm{dom}(f)$ there is a $g \in F$ with $x \in \mathrm{dom}(g)$ and $g(x) = f(x)$. So $f \sqsubseteq \bigsqcup F$.
End answer . . . .

4. Let X be a set and let $\wp(X)$ be the power set of X ordered by inclusion $\subseteq$.

(a) Describe the compact elements of $\wp(X)$ and prove that these are indeed the compact elements..

Answer: . . . .
If $Y \subseteq X$ and Y is **infinite**, it is not compact: the set

$$Y_{\mathrm{fin}} := \{Z \mid Z \subseteq Y, Z \text{ finite}\}$$

is a directed set and $\bigcup Y_{\mathrm{fin}} \stackrel{*}{=} Y$, but there is no $Z \in Y_{\mathrm{fin}}$ such that $Y \subseteq Z$.
NB1. Y_{fin} is directed because if Z_1 and Z_2 are elements of Y_{fin}, then

$Z_1 \cup Z_2$ is also an element of Y and $Z_1, Z_2 \subseteq Z_1 \cup Z_2$.
NB2. For $\bigcup Y_{\text{fin}} \stackrel{*}{=} Y$: If $y \in \bigcup Y_{\text{fin}}$, then $y \in Z \subseteq Y$ for some Z, so $y \in Y$. If $y \in Y$, then $y \in \{y\}$ and $\{y\} \in Y_{\text{fin}}$, so $y \in \bigcup Y_{\text{fin}}$.
If $Y \subseteq X$ and Y is **finite**, it is compact: suppose $Y = \{y_1, \ldots, y_n\}$ and let W be a directed set of subsets with $\bigcup W = Y$. Then for every y_i there is a $Z_i \in W$ with $y_i \in Z_i$. As W is directed, there is a $Z \in W$ with $Z_1 \cup \ldots \cup Z_n \subseteq Z$. But then $Y \subseteq Z$.
So, the *basis* of the domain $\wp(X)$ is

$$\mathbf{B}_{\wp(X)} = \{Y \mid Y \subseteq X \text{ with } Y \text{ finite}\}$$

and note that if X is finite, then $\mathbf{B}_{\wp(X)} = \wp(X)$ (all elements are compact).
End answer...

(b) Show that $\wp(X)$ is an algebraic dcpo.
Answer: ....
Given $Y \subseteq X$, consider $Y_{\text{fin}} := \{Z \mid Z \subseteq Y, Z \in \mathbf{B}_{\wp(X)}\}$. Then

$$Y_{\text{fin}} = \{Z \mid Z \subseteq Y, Z \text{ finite}\}$$

and we need to show that $Y = \bigcup Y_{\text{fin}}$, which we have done in the answer to Exercise 4a: see the NB2.
End answer...

5. Suppose we are in a dcpo where each pair of elements has a glb. Show that $\forall x, y (x \sqsubseteq y \Leftrightarrow x = x \sqcap y)$.

6. (a) Prove that $\mathbb{N}_\perp$ satisfies (axiom d):

$$\forall x, y, z \in \mathbb{N}_\perp (y \uparrow z \Rightarrow x \sqcap (y \sqcup z) = (x \sqcap y) \sqcup (x \sqcap z)).$$

   (b) Prove that $\mathbb{N}_\perp \to \mathbb{N}_\perp$ (the set of Scott continuous functions) satisfies (axiom d).

7. We are in a bounded complete p.o. and we consider the property $(*)$ (used in stability)

$$\forall x, y \in D(x \uparrow y \to f(x \sqcap y) = f(x) \sqcap f(y)) \qquad (*)$$

   Show that, if f satisfies $(*)$, then it is monotone.

8. (a) Define all possible different "AND" functions as monotone functions (in $\mathbb{B}_\perp \times \mathbb{B}_\perp \to \mathbb{B}_\perp$).
   (b) Show that 3 of your functions can be defined in PCF.
   (c) Now show that one of your functions cannot be defined in PCF
      i. by semantic means (using dI-domains).
      ii. by using the non-definability of por

9. Prove that the identity function $I : \mathbb{N}_\perp \to \mathbb{N}_\perp$ is not very finite.

10. Prove that, in a bounded complete dcpo, every non-empty set X has a greatest lower bound, $\bigsqcap X$.