Semantics and Domain theory
Exercises 10
Exercises from the dI-domains notes

1. Suppose that a monotonic function \( p : (B_\bot \times B_\bot) \to B_\bot \) satisfies
   \[
   \begin{align*}
   p(tt, \bot) &= \text{tt}, \\
   p(\bot, tt) &= \text{tt}, \\
   p(ff, ff) &= ff.
   \end{align*}
   \]
   Show that \( p \) coincides with the parallel-or function (on Slide 72 of the notes of Pitts) in the sense that
   \[ p(d_1, d_2) = \text{por}(d_1)(d_2), \]
   for all \( d_1, d_2 \in B_\bot \).

2. Show that the evaluation relation for PCF+por (Slide 77, where rules for \( \text{por}(M_1, M_2) \Downarrow V \) have been added to PCF) is still deterministic: If \( M \Downarrow V \), then for all \( V' \), if \( M \Downarrow V' \), then \( V = V' \). This is again proved by induction on the derivation of \( M \Downarrow V \); do the case for the new \( \text{por} \)-rules.

3. (a) Describe the compact elements of \( N_\bot \to N_\bot \) and prove that these are indeed the compact elements.
   (b) Show that \( N_\bot \to N_\bot \) is an algebraic dcpo.

4. Let \( X \) be a set and let \( \wp(X) \) be the power set of \( X \) ordered by inclusion \( \subseteq \).
   (a) Describe the compact elements of \( \wp(X) \) and prove that these are indeed the compact elements.
   (b) Show that \( \wp(X) \) is an algebraic dcpo.

5. Suppose we are in a dcpo where each pair of elements has a glb. Show that
   \[ \forall x, y(x \sqsubseteq y \iff x = x \sqcap y). \]

   \textbf{Answer:} ..............................................................
   
   We always have \( x \sqcap y \sqsubseteq x \). If \( x \sqsubseteq y \), then \( x \sqsubseteq y \land x \sqsubseteq x \), so \( x \sqsubseteq x \sqcap y \), so we have \( x = x \sqcap y \).
   
   If \( x = x \sqcap y \) then from \( x \sqcap y \sqsubseteq y \) we conclude \( x \sqsubseteq y \).

   \textbf{End answer} ..............................................................

6. (a) Prove that \( N_\bot \) satisfies (axiom d):
   \[ \forall x, y, z \in N_\bot (y \uparrow z \Rightarrow x \sqcap (y \sqcup z) = (x \sqcap y) \sqcup (x \sqcap z)). \]

   \textbf{Answer:} ..............................................................
   
   Assume \( y \uparrow z \). This means we have \( 1 \) \( y = z \) or \( 2 \) \( y = \bot, z \in N \) or \( 3 \) \( y \in N, z = \bot \).
   In case \( 1 \) we have \( x \sqcap (y \sqcup z) = (x \sqcap y) \sqcup (x \sqcap z) \), which holds.
   In case \( 2 \) we have \( x \sqcap z = \bot \sqcup (x \sqcap z) \), which holds.
   Case \( 3 \) is similar to case \( 2 \).

   \textbf{End answer} ..............................................................

   (b) Prove that \( N_\bot \to N_\bot \) (the set of Scott continuous functions) satisfies
   (axiom d).

   \textbf{Answer:} ..............................................................
   
   First observe the following.
• $g \uparrow h$ means $\forall x \in \text{dom}(g) \cap \text{dom}(h)(g(x) = h(x))$.
• If $g \uparrow h$, $g \sqcup h$ is the function
  \[
  (g \sqcup h)(x) = \begin{cases} 
  g(x) & \text{if } g(x) \neq \bot \\
  h(x) & \text{if } h(x) \neq \bot \\
  \bot & \text{otherwise.}
  \end{cases}
  \]
• $f \sqcap g$ is the function
  \[
  (f \sqcap g)(x) = \begin{cases} 
  f(x) & \text{if } x \in \text{dom}(f) \cap \text{dom}(g) \land f(x) = g(x) \\
  \bot & \text{otherwise.}
  \end{cases}
  \]

Now assume $g \uparrow h$. We have
\[
(f \sqcap (g \sqcup h))(x) = \begin{cases} 
  f(x) & \text{if } x \in \text{dom}(f) \cap \text{dom}(g \sqcup h) \land f(x) = (g \sqcup h)(x) \\
  \bot & \text{otherwise.}
  \end{cases}
\]

We also have
\[
((f \sqcap g) \sqcup (f \sqcap h))(x) = \begin{cases} 
  (f \sqcap g)(x) & \text{if } (f \sqcap g)(x) \neq \bot \\
  (f \sqcap h)(x) & \text{if } (f \sqcap h)(x) \neq \bot \\
  \bot & \text{otherwise.}
  \end{cases}
\]

\[
= \begin{cases} 
  f(x) & \text{if } x \in \text{dom}(f) \cap \text{dom}(g) \land f(x) = g(x) \\
  f(x) & \text{if } x \in \text{dom}(f) \cap \text{dom}(h) \land f(x) = h(x) \\
  \bot & \text{otherwise.}
  \end{cases}
\]

\[
= \begin{cases} 
  f(x) & \text{if } (f(x) = g(x) \neq \bot) \lor f(x) = h(x) \neq \bot \\
  \bot & \text{otherwise.}
  \end{cases}
\]

So $f \sqcap (g \sqcup h) = (f \sqcap g) \sqcup (f \sqcap h)$.

**End answer**

7. We are in a bounded complete p.o. and we consider the property $(\ast)$ (used in stability)

\[
\forall x, y \in D(x \uparrow y \rightarrow f(x \sqcap y) = f(x) \sqcap f(y))
\]

Show that, if $f$ satisfies $(\ast)$, then it is monotone.

**Answer:**

Suppose $x \sqsubseteq y$. By Exercise 5 we have $x = x \sqcap y$. Also $x \uparrow y$, so we have:

\[
f(x) = f(x \sqcap y) = f(x) \sqcap f(y).
\]

So $f(x) = f(x) \sqcap f(y)$ and again by Exercise 5 we conclude that $f(x) \sqsubseteq f(y)$.

**End answer**

8. (a) Define all possible different “AND” functions as monotone functions (in $\mathbb{B}_1 \times \mathbb{B}_1 \rightarrow \mathbb{B}_1$).

**Answer:**

There are 4 different monotone “AND” functions. For each and function we have

- and(tt, tt) = tt,
• and(tt, ff) = and(ff, tt) = and(ff, ff) = ff,
• and(⊥, ⊥) = and(tt, ⊥) = and(⊥, tt) = ⊥.

Then we differentiate between them as follows.

i. and₁(ff, ⊥) = ff, and₁(⊥, ff) = ⊥.
ii. and₂(ff, ⊥) = ⊥, and₂(⊥, ff) = ff.
iii. and₃(ff, ⊥) = ⊥, and₃(⊥, ff) = ⊥.
iv. and₄(ff, ⊥) = ff, and₄(⊥, ff) = ff.

End answer

(b) Show that 3 of your functions can be defined in PCF.

Answer:

i. and₁ is defined by $fn \ x : bool. \ fn \ y : bool. \ if \ x \ then \ y \ else \ false$.

Compute for yourself that this indeed defines and₁.

ii. and₂ is defined by $fn \ x : bool. \ fn \ y : bool. \ if \ y \ then \ x \ else \ false$.

Compute for yourself that this indeed defines and₂.

iii. and₃ is defined by $fn \ x : bool. \ fn \ y : bool. \ if \ x \ then \ y \ else \ (if \ y \ then \ false \ else \ false)$.

Compute for yourself that this indeed defines and₃.

End answer

(c) Now show that one of your functions cannot be defined in PCF

i. by semantic means (using dI-domains).

Answer: 

The function and₄ is not stable, and hence it cannot be defined. That and₄ is not stable is seen as follows. We have $(ff, ⊥) ↑ (⊥, ff)$, but

$$ff = and₄(ff, ⊥) ⊓ and₄(⊥, ff) \neq and₄((ff, ⊥) ⊓ (⊥, ff)) = ⊥.$$

End answer

ii. by using the non-definability of por

Answer: 

The function and₄ cannot be defined, because if it could, say by the term $M : bool → bool → bool$, then we would also be able to define por (parallel or), by

$$P := fn \ x : bool. \ fn \ y : bool. \ neg(M(neg \ x)(neg \ y)),$$

where $neg := fn \ x : bool. \ if \ x \ then \ false \ else \ true$ defines negation. (Check this by computing the semantics of $P$.)

End answer

9. Prove that the identity function $I : N_⊥ → N_⊥$ is not very finite.

Answer: 

Consider, for $i ∈ N$,

$$f_i(x) := \begin{cases} x & \text{if } x ≤ i \\ ⊥ & \text{if } x = ⊥ \text{ or } x > i \end{cases}$$

We have $f_i ⊑ I$ and all the $f_i$ are different, so $\{g | g ⊑ I\}$ is not finite, so $I$ is not very finite.

End answer
10. Prove that, in a bounded complete dcpo $D$, every non-empty set $X$ has a greatest lower bound, $\bigsqcap X$.

**Answer:** .................................................................

Given $X \neq \emptyset$, there is an $x_0 \in X$, so $Y := \{d \in D \mid \forall x \in X (d \sqsubseteq x)\}$ is bounded (by $x_0$), so we can define

$$d_0 := \bigsqcup \{d \in D \mid \forall x \in X (d \sqsubseteq x)\}.$$  

We claim that $d_0 = \bigsqcap X$. We need to prove the following two properties.

*(glb1)* To prove: $\forall x \in X (d_0 \sqsubseteq x)$.

Suppose $x \in X$. Then $\forall d \in Y (d \sqsubseteq x)$, so $x$ is an upperbound of $Y$, so $d_0 \sqsubseteq x$, because $d_0$ is the least upperbound of $Y$.

*(glb2)* To prove: $\forall d (\forall x \in X (d \sqsubseteq x)) \Rightarrow d \sqsubseteq d_0$.

Suppose $\forall x \in X (d \sqsubseteq x)$. Then $d \in Y$, so $d \sqsubseteq d_0$, because $d_0$ is an upperbound of $Y$.

End answer.................................................................