Test exam **Semantics and Domain Theory** ?? ?? June 2013, ?? –??

The maximum number of points per question is given in the margin. (Maximum 100 points in total.)

**NB: you can use all your notes and course material**

1. We define the program

   \[ S := \text{while } x > 0 \text{ do } (x := x - 1; y := y + z) \]

   and we define \( w_n : \text{State } \rightarrow \text{State} \) as follows: \( w_0(s) = \bot \) and for \( n > 0, \)

   \[
   w_n(s) := \begin{cases} 
   s & \text{if } s(x) \leq 0 \\
   s[x \mapsto 0, y \mapsto s(y) + s(x) \ast s(z)] & \text{if } 0 < s(x) < n \\
   \bot & \text{if } s(x) \geq n 
   \end{cases}
   \]

   (10) Determine the functional \( F \) that we need in order to compute the denotational semantics of \( S \) and show that \( F(w_n) = w_{n+1} \)

   (5) Determine \( w_\infty := \bigsqcup_{i\in\mathbb{N}} w_n \) and show that \( w_\infty \) is a fixed-point of \( F \).

2. Show that, for \((D, \sqsubseteq)\) a cpo, \((D, \sqsubseteq) \xrightarrow{\text{mon}} (D, \sqsubseteq)\) is also a cpo.
   \((D, \sqsubseteq) \xrightarrow{\text{mon}} (D, \sqsubseteq)\) is the set of *monotone* functions from \((D, \sqsubseteq)\) to \((D, \sqsubseteq)\).

   (10) Let \( p \in \mathbb{N} \). Given a sequence of sets \( D_0 \subseteq D_1 \subseteq \ldots D_i \ldots \subseteq \mathbb{N} \), define the sequence of functions \( k_i : \mathbb{N}_\bot \xrightarrow{\text{mon}} \mathbb{N}_\bot \) by

   \[
   \begin{cases} 
   k_i(x) := p & \text{if } x \in D_i \\
   k_i(x) := \bot & \text{if } x = \bot \text{ or } x \in \mathbb{N} \setminus D_i 
   \end{cases}
   \]

   Show that \((k_i)_{i\in\mathbb{N}}\) is a chain.

   (b) Consider the function \( H : (\mathbb{N}_\bot \xrightarrow{\text{mon}} \mathbb{N}_\bot) \rightarrow [0, 1] \) defined as follows

   \[
   H(f) := \begin{cases} 
   0 & \text{if } \forall x \in \mathbb{N}_\bot (f(x) = \bot) \\
   \frac{1}{n+1} & \text{if } n \text{ is the smallest } n \text{ for which } f(n) \neq \bot 
   \end{cases}
   \]

   where we mean *smallest* with respect to the ordinary \( \leq \)-relation on \( \mathbb{N} \).

   **NB.** \([0, 1]\) is just the closed interval from 0 to 1; the ordering on \([0, 1]\) is just the well-known one \( \leq \).
i. Prove that $H$ is monotone.

ii. Compute $H(\sqcup_{i \in \mathbb{N}} k_i)$.

iii. Prove that $H$ is continuous.

4. Define the sequence of functions $f_i : \mathbb{N}_\bot \times \mathbb{N}_\bot \to \mathbb{N}_\bot$ as follows ($n$ and $m$ range over $\mathbb{N}$).

\[
\begin{cases}
  f_i(x, y) := \bot & \text{if } x = \bot \text{ or } y = \bot \\
  f_i(n, m) := k & \text{if } k \text{ is the smallest } k \leq i \text{ such that } k \cdot n \geq m \\
  f_i(n, m) := \bot & \text{if } \forall k \leq i (k \cdot n < m)
\end{cases}
\]

(a) Show that each $f_i$ is monotone and continuous. 

(b) Show that $(f_i)_{i \in \mathbb{N}}$ is a chain.

(c) Compute $\bigcup_{i \in \mathbb{N}} f_i$.

5. We give the following interpretation of the less-than-or-equal ordering on the natural numbers as a domain-theoretic function $\preceq : \mathbb{N}_\bot \to \mathbb{N}_\bot \to \mathbb{B}_\bot$, where $\mathbb{B}_\bot$ is the well-known flat cpo of booleans.

\[
\begin{array}{c|c|c|c}
\preceq & \bot & n \\
\hline
\bot & \bot & \bot \\
\hline
m & \text{tt if } n \leq m & \text{ff if } n > m \\
\end{array}
\]

(a) Prove that $\preceq$ is monotone and continuous.

(b) Give a different interpretation of the less-than-or-equal ordering on the natural numbers as a continuous domain-theoretic function $\preceq : \mathbb{N}_\bot \to \mathbb{N}_\bot \to \mathbb{B}_\bot$, and prove that it is monotone.

6. (a) Prove that the denotational semantics of the following PCF terms are the same: $M := \textbf{fix} (\text{fn } f : \tau \to \tau.f)$ and $N := \text{fn } y : \tau.\textbf{fix} (\text{fn } x : \tau.x)$.

(b) Prove that the denotational semantics of the following PCF terms, $S$ and $T$, are the same. (Here, $p, q$ and $r$ are boolean expressions, $K, L$ are arbitrary expressions of the same type.)

\[
\begin{align*}
S & := \text{if } (\text{if } p \text{ then } q \text{ else } r) \text{ then } K \text{ else } L \\
T & := \text{if } p \text{ then } (\text{if } q \text{ then } K \text{ else } L) \text{ else } (\text{if } r \text{ then } K \text{ else } L).
\end{align*}
\]
7. Show the following two cases in the proof of the substitution property (slide 38):
if \( M_1 \) then \( M_2 \) else \( M_3 \) and \( \text{fn } x : \tau. M \).

8. Prove Lemma 7.2.1 (iii) in the course notes of Winskel.

9. (a) Show that, if the number of elements in the cpo \( D \) is finite and \( |D| \geq 2 \) then \( |[D \to D]| > |D| \). (That is: there are strictly more continuous functions on \( D \) than elements of \( D \).)

   (b) Prove (using the result under (a)) that a \( \lambda \)-model cannot be finite (unless it is trivial and contains only one element).

10. Let \( M_1 \), \( M_2 \) and \( M_3 \) be \( \lambda \)-terms that satisfy the following equations

\[
M_1 = (\lambda x. \lambda y. y x) M_1 \\
M_2 = (\lambda x. \lambda y. y (x y)) M_2 \\
M_3 = (\lambda x. \lambda y. y M_3) M_3.
\]

For which \( i, j \) do we have \( D_A \models M_i = M_j \)? Prove your answer.

11. (a) In the model \( D_A \), \( F \circ G = \text{id}_{[D \to D]} \), but \( G \circ F \neq \text{id}_D \).

   Is it the case (in \( D_A \)) that \( G \circ F \subseteq \text{id}_D \)? Or \( G \circ F \subseteq \text{id}_D \)? Or is neither the case?

   (b) Show in detail that, in the model \( D_A \), \( F \) is continuous in its first argument.