Semantics and Domain theory

Exercises 7

1. Show that, if $Q \Downarrow_\tau V$, then $(\mathbf{fn}\, x : \tau.\, \mathbf{fn}\, y : \tau.y)PQ \Downarrow V$. (NB. V denotes an arbitrary *value*.)

2. The proof that PCF evaluation is deterministic is by induction on the derivation.
   (In the notes, Proposition 5.4.1, this is proved by showing that the following set is closed under the rules of Fig.3:
   $\{(M, \tau, V) | M \Downarrow_\tau V \wedge \forall V'(M \Downarrow_\tau V' \Rightarrow V = V')\}$.)

   An alternative way (and imho an easier way) of looking at this is to prove the following:
   $$M \Downarrow_\tau V \Rightarrow \forall V'(M \Downarrow_\tau V' \Rightarrow V = V')$$
   by induction on the derivation of $M \Downarrow_\tau V$. Do only the cases when the last applied rule is $(\Downarrow_{\text{if1}})$ or $(\Downarrow_{\text{cbn}})$.

3. Prove that the following terms M and N are not contextually equivalent.
   (a) $M = \mathbf{if}\ x\ \mathbf{then}\ 0\ \mathbf{else}\ 1$ and $N = \mathbf{if}\ y\ \mathbf{then}\ 0\ \mathbf{else}\ 1$.
   (b) $M = \mathbf{fn}\, x : \mathbf{nat}.\mathbf{succ}(\mathbf{pred}\, x)$ and $N = \mathbf{fn}\, x : \mathbf{nat}.x$.

4. (a) Give a type τ, a term M, values V, V' and a context $C[-]$ such that $M \Downarrow_\tau V$ but $C[M] \Downarrow_\tau V' \neq C[V]$.
   (b) Give a type τ, a term M, a value V and a context $C[-]$ such that $M \Downarrow_\tau V$ but $C[M] \not\Downarrow_\tau$ ($C[M]$ has no value.)
   (c) Give a type τ, a term M, a value V and a context $C[-]$ such that $M \not\Downarrow_\tau$ but $C[M] \Downarrow_\tau V$

5. Given the definition of plus (Exercise 5.6.3.)
   $$\begin{aligned}\text{plus} \quad = \quad & \mathbf{fix}(\mathbf{fn}\, p : \mathbf{nat} \to \mathbf{nat} \to \mathbf{nat}.\, \mathbf{fn}\, x : \mathbf{nat}.\, \mathbf{fn}\, y : \mathbf{nat}. \\ & \ \mathbf{if}\ \mathbf{zero}(y)\ \mathbf{then}\ x\ \mathbf{else}\ \mathbf{succ}(p\ x\ \mathbf{pred}(y)))\end{aligned}$$
   Prove (by induction) that
   $$\forall m, n(\text{plus}\ \mathbf{succ}^m(0)\ \mathbf{succ}^n(0)\ \Downarrow_{\mathbf{nat}}\ \mathbf{succ}^{m+n}(0))$$
   NB. First identify the proper statement that you need and that you can prove relatively easily by induction.