Semantics and Domain theory

Exercises 8

1. (Exercise 6.5.2. of Pitts' notes) Define $\Omega_\tau = \mathbf{fix}(\mathbf{fn}\, x : \tau . x)$
   (a) Show that $[\![\Omega_\tau]\!]$ is the least element of the domain $[\![\tau]\!]$.
   (b) Deduce that $[\![\mathbf{fn}\, x : \tau . \Omega_\tau]\!] = [\![\Omega_{\tau \to \tau}]\!]$.

2. (a) Compute the denotational semantics of
   $M = \mathbf{fn}\, x : \mathbf{bool}.\, \mathbf{fn}\, y : \mathbf{nat}.\mathbf{if}\ x\ \mathbf{then}\ y\ \mathbf{else}\ y$.
   (b) Define a term P such that $[\![M]\!] \sqsubseteq [\![P]\!]$ but $[\![M]\!] \neq [\![P]\!]$.
   (c) Define a term N such that $[\![M]\!] = [\![N]\!]$ but $N \not\Downarrow M$.

3. Define terms $M, N : \mathbf{nat} \to \mathbf{nat}$ with $[\![M]\!] \sqsubseteq [\![N]\!]$ and $[\![M]\!] \neq [\![N]\!]$.

4. Verify that $[\![(\mathbf{fn}\, x : \sigma . M) N]\!](\rho) = [\![M[N/x]]\!](\rho)$ for M, N with $\Gamma \vdash N : \sigma$ and $\Gamma, x : \sigma \vdash M : \tau$ and $\rho \in [\![\Gamma]\!]$. (Use the result on Slide 62, the Substitution Lemma.)

5. Prove Soundness of the operational semantics (Theorem 6.4.1) for the inductive cases $\Downarrow_{\text{pred}}$ and $\Downarrow_{\text{if1}}$.
   Remember that Theorem 6.4.1 states that for all closed expressions M and V and type τ, if $M \Downarrow_\tau V$, then $[\![M]\!] = [\![V]\!]$. It is proved by induction on the derivation of $M \Downarrow_\tau V$.

6. Let

   $$P := \mathbf{fix}(\mathbf{fn}\, p : \mathbf{nat} \to \mathbf{bool}.\, \mathbf{fn} x : \mathbf{nat}.\, \mathbf{if}\ (\mathbf{zero}\, x)\ \mathbf{then\ true\ else}\ p(\mathbf{pred}\,(\mathbf{pred}\, x))).$$

   Compute $[\![P]\!]$.

7. We give PCF-terms that define the "or" function, taking partiality into account. A PCF-term M *defines* the domain-theoretic function f if $[\![M]\!] = f$.
   (a) Give a PCF-term that defines the function $f : \mathbb{B}_\perp \to \mathbb{B}_\perp \to \mathbb{B}_\perp$ given by

   $$f(x)(y) := \begin{cases} \perp & \text{if } x = \perp \text{ or } (x = \text{ff and } y = \perp) \\ \text{tt} & \text{if } x = \text{tt or } (x = \text{ff and } y = \text{tt}) \\ \text{ff} & \text{if } x = \text{ff and } y = \text{ff} \end{cases}$$

   (b) Give a PCF-terms that defines the function $g : \mathbb{B}_\perp \to \mathbb{B}_\perp \to \mathbb{B}_\perp$ given by

   $$g(x)(y) := \begin{cases} \perp & \text{if } x = \perp \text{ or } y = \perp \\ \text{tt} & \text{if } x, y \neq \perp \text{ and } (x = \text{tt or } y = \text{tt}) \\ \text{ff} & \text{if } x = \text{ff and } y = \text{ff} \end{cases}$$