RADBOUD UNIVERSITY NIJMEGEN

Science Faculty

Test exam **Semantics and Domain Theory** ?? ?? June 2013, ?? –??
The maximum number of points per question is given in the margin. (Maximum 100 points in total.)
NB: you can use all your notes and course material

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1. We define the program

$$S := \textbf{while}\, x > 0\, \textbf{do}\, (x := x - 1; y := y + z)$$

and we define w_n : State $\rightharpoonup$ State as follows: $w_0(s) = \bot$ and for $n > 0$,

$$w_n(s) := \begin{cases} s & \text{if } s(x) \leq 0 \\ s[x \mapsto 0, y \mapsto s(y) + s(x) * s(z) & \text{if } 0 < s(x) < n \\ \bot & \text{if } s(x) \geq n \end{cases}$$

(10) (a) Determine the functional F that we need in order to compute the denotational semantics of S and show that $F(w_n) = w_{n+1}$

(5) (b) Determine $w_\infty := \sqcup_{i \in \mathbb{N}} w_n$ and show that w_∞ is a fixed-point of F.

(10) 2. Show that, for $(D, \sqsubseteq)$ a cpo, $(D, \sqsubseteq) \stackrel{\text{mon}}{\to} (D, \sqsubseteq)$ is also a cpo.
$((D, \sqsubseteq) \stackrel{\text{mon}}{\to} (D, \sqsubseteq)$ is the set of *monotone* functions from $(D, \sqsubseteq)$ to $(D, \sqsubseteq)$.)

(10) 3. (a) Let $p \in \mathbb{N}$. Given a sequence of sets $D_0 \subseteq D_1 \subseteq \ldots D_i \ldots \subseteq \mathbb{N}$, define the sequence of functions $k_i : \mathbb{N}_\bot \stackrel{\text{mon}}{\to} \mathbb{N}_\bot$ by

$$\begin{cases} k_i(x) := p & \text{if } x \in D_i \\ k_i(x) := \bot & \text{if } x = \bot \text{ or } x \in \mathbb{N} \setminus D_i \end{cases}$$

Show that $(k_i)_{i \in \mathbb{N}}$ is a chain.

(20) (b) Consider the function $H : (\mathbb{N}_\bot \stackrel{\text{mon}}{\to} \mathbb{N}_\bot) \to [0, 1]$ defined as follows

$$H(f) := \begin{cases} 0 & \text{if } \forall x \in \mathbb{N}_\bot (f(x) = \bot) \\ \frac{1}{n+1} & \text{if } n \text{ is the smallest } n \text{ for which } f(n) \neq \bot \end{cases}$$

where we mean *smallest* with respect to the ordinary $\leq$-relation on $\mathbb{N}$.
NB. $[0, 1]$ is just the closed interval from 0 to 1; the ordering on $[0, 1]$ is just the well-known one $\leq$.

i. Prove that H is monotone.

ii. Compute $H(\sqcup_{i\in\mathbb{N}} k_i)$.

iii. Prove that H is continuous.

4. Define the sequence of functions $f_i : \mathbb{N}_\perp \times \mathbb{N}_\perp \to \mathbb{N}_\perp$ as follows (n and m range over $\mathbb{N}$).

$$\begin{cases} f_i(x,y) := \perp & \text{if } x = \perp \text{ or } y = \perp \\ f_i(n,m) := k & \text{if } k \text{ is the smallest } k \leq i \text{ such that } k * n \geq m \\ f_i(n,m) := \perp & \text{if } \forall k \leq i (k * n < m) \end{cases}$$

(10) (a) Show that each f_i is monotone and continuous.

(10) (b) Show that $(f_i)_{i\in\mathbb{N}}$ is a chain.

(10) (c) Compute $\cup_{i\in\mathbb{N}} f_i$.

5. We give the following interpretation of the less-than-or-equal ordering on the natural numbers as a domain-theoretic function $\preceq: \mathbb{N}_\perp \to \mathbb{N}_\perp \to \mathbb{B}_\perp$, where $\mathbb{B}_\perp$ is the well-known flat cpo of booleans.

$\preceq$	$\perp$	n
$\perp$	$\perp$	$\perp$
m	$\perp$	**tt** if $n \leq m$ **ff** if $n > m$

(5) (a) Prove that $\preceq$ is monotone and continuous.

(10) (b) Give a different interpretation of the less-than-or-equal ordering on the natural numbers as a continuous domain-theoretic function $\preceq: \mathbb{N}_\perp \to \mathbb{N}_\perp \to \mathbb{B}_\perp$, and prove that it is monotone.

(5) 6. (a) Prove that the denotational sematics of the following PCF terms are the same: $M := \mathbf{fix}\,(\mathbf{fn}\, f : \tau \to \tau . f)$ and $N := \mathbf{fn}\, y : \tau . \mathbf{fix}\,(\mathbf{fn}\, x : \tau . x)$.

(5) (b) Prove that the denotational sematics of the following PCF terms, S and T, are the same. (Here, p, q and r are boolean expressions, K, L are arbitrary expressions of the same type.)

$$\begin{aligned} S &:= \mathbf{if}\,(\,\mathbf{if}\, p\, \mathbf{then}\, q\, \mathbf{else}\, r)\, \mathbf{then}\, K\, \mathbf{else}\, L \\ T &:= \mathbf{if}\, p\, \mathbf{then}\,(\,\mathbf{if}\, q\, \mathbf{then}\, K\, \mathbf{else}\, L)\, \mathbf{else}\,(\,\mathbf{if}\, r\, \mathbf{then}\, K\, \mathbf{else}\, L). \end{aligned}$$

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(10) 7. Show the following two cases in the proof of the substitution property (slide 38): **if** M_1 **then** M_2 **else** M_3 and **fn** $x : \tau.M$.

(10) 8. Prove Lemma 7.2.1 (iii) in the course notes of Winskel.

(10) 9. (a) Show that, if the number of elements in the cpo D is finite and $|D| \geq 2$ then $|[D \to D]| > |D|$. (That is: there are strictly more continuous functions on D then elements of D.)

(10) (b) Prove (using the result under (a)) that a λ-model cannot be finite (unless it is trivial and contains only one element).

(15) 10. Let M_1, M_2 and M_3 be λ-terms that satisfy the following equations

$$\begin{aligned} M_1 &= (\lambda x.\, \lambda y.\, y\, x)\, M_1 \\ M_2 &= (\lambda x.\, \lambda y. y\, (x\, y))\, M_2 \\ M_3 &= (\lambda x.\, \lambda y.\, y\, M_3)\, M_3. \end{aligned}$$

For which i, j do we have $D_A \models M_i = M_j$? Prove your answer.

(10) 11. (a) In the model D_A, $F \circ G = \mathrm{id}_{[D \to D]}$, but $G \circ F \neq \mathrm{id}_D$. Is it the case (in D_A) that $G \circ F \sqsubseteq \mathrm{id}_D$? Or $G \circ F \sqsubseteq \mathrm{id}_D$? Or is neither the case?

(10) (b) Show in detail that, in the model D_A, F is continuous in its first argument.

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