Semantics and Domain theory

Exercises 12

1. We consider the model definition as explained in the lecture. (See Definition 57 of Berline; so we assume that the interpretation $[\![-]\!]_\rho$ is well-defined.) Assume $G \circ A = \mathrm{id}_M$. Show that the η-rule holds in the model. (The η-rule says: $\lambda x.N\,x = N$ if $x \notin \mathrm{FV}(N)$.) NB. You may need to use the follwoing property (without proof): If $\rho(x) = \rho'(x)$ for all $x \in \mathrm{FV}(M)$, then $[\![M]\!]_\rho = [\![M]\!]_{\rho'}$.

2. Which of the following sets are complete lattices.

   (a) The set of flat natural numbers $\mathbb{N}_\bot$.

   (b) The set $\mathcal{P}_{\mathrm{fin}}(\mathbb{N})$ of *finite subsets* of $\mathbb{N}$.

   (c) The set Ω (= $\mathbb{N} \cup \{\omega\}$, with the ordering we have seen before).

   (d) The set of monotone functions from $\mathbb{B}_\bot^\top$ to $\mathbb{B}_\bot^\top$.
   (Remember that the set of flat booleans with a top element added, $\mathbb{B}_\bot^\top$, is a complete lattice.)

3. Complete the proof of Proposition 3.1.7.
   That is, show that in a complete lattice $(D, \sqsubseteq)$, if we define

   $$\bigsqcap X := \bigsqcup \{y \in D \mid y \sqsubseteq X\},$$

   then $\bigsqcap X$ is indeed the *greatest lower bound* (also called the *inf*) of X.

4. Prove the correctness of Definition 3.2.5. To prove this, you have to show that the function

   $$\boldsymbol{\lambda} d.[\![P]\!]_{\rho(x:=d)}$$

   is continuous for every P and ρ. (You may assume that F and G are continuous and all the other results about continuity from the notes.)

5. At the lecture, we have seen the interpretations in D_A of **I** (= $\lambda x.x$), **K** (= $\lambda x.\lambda y.x$) and **II**.

   (a) Compute the interpretation of $\lambda x.x\,x$.

   (b) Show that $[\![\mathbf{KI}]\!] = \{(\beta, (\gamma, c)) \mid c \in \gamma\}$ (without doing a β-reduction first).

6. Let Y be an element of D_A and let ρ be a valuation with $\rho(y) = Y$.

   (a) Compute in D_A the interpretation of $\lambda x.y\,x$ by expressing $[\![\lambda x.y\,x]\!]_\rho$ in terms of Y.

   (b) Conclude that the η-rule does not hold in D_A. (The η-rule says that $\lambda x.M\,x = M$ if $x \notin \mathrm{FV}(M)$.)

7. Use the result of the following exercise ($[\![\Omega]\!] = \emptyset$) to

   (a) compute the interpretation of $\lambda y.\Omega$ in D_A,

   (b) compute the interpretation of $\lambda y.y\,\Omega$ in D_A.

8. [Challenging] Show that the interpretation of Ω (= $(\lambda x.x\,x)(\lambda x.x\,x)$) in D_A is $\emptyset$.
   (Hint: From a $c \in [\![\Omega]\!]$ you can construct an infinite sequence $(\alpha_i)_{i \in \mathbb{N}}$ with $(\alpha_{i+1}, c) \in \alpha_i$ for all i, which is impossible in D_A.)