Semantics and Domain theory

Exercises 2

NB. We write $\mathsf{State} \rightharpoonup \mathsf{State}$ for the set of partial functions from State to State.

1. Define the denotational semantics of **repeat** P **until** b as a fixed point of a function $g : (\mathsf{State} \rightharpoonup \mathsf{State}) \to (\mathsf{State} \rightharpoonup \mathsf{State})$.
   (NB this program executes statement P and then checks the boolean b; if b holds, execution stops, if b doesn't hold, it iterates.)

2. [Exercise 4.2 of Nielsen & Nielsen] Consider the statement

$$S := \textbf{ while } x \neq 0 \textbf{ do } x := x - 1$$

   (a) Determine the functional $F : (\mathsf{State} \rightharpoonup \mathsf{State}) \to (\mathsf{State} \rightharpoonup \mathsf{State})$ associated with this statement. (The F we need to take the fixed point of to determine the semantics of S.)

   (b) Determine for each of the following partial functions $g : \mathsf{State} \rightharpoonup \mathsf{State}$ whether it is a fixed point of F.

   - $g_1(s) := \uparrow$ for all $s \in \mathsf{State}$
   - $g_2(s) := \begin{cases} s[x \mapsto 0] & \text{if} \quad s(x) \geq 0 \\ \uparrow & \text{if} \quad s(x) < 0 \end{cases}$
   - $g_3(s) := \begin{cases} s[x \mapsto 0] & \text{if} \quad s(x) \geq 0 \\ s & \text{if} \quad s(x) < 0 \end{cases}$
   - $g_4(s) := s[x \mapsto 0]$ for all $s \in \mathsf{State}$
   - $g_5(s) := s$ for all $s \in \mathsf{State}$

   (c) Which of the above (if any) is the least fixed point of F?

3. Consider the function $f : (\mathsf{State} \rightharpoonup \mathsf{State}) \to (\mathsf{State} \rightharpoonup \mathsf{State})$ as defined by Pitts on slide 12, but now with $\mathsf{State} = \mathbb{L} \to \mathbb{Z}$:

$$f(w)(s) := \begin{cases} s & \text{if } s(x) \leq 0 \\ w(s[x \mapsto s(x) - 1, y \mapsto s(x) * s(y)]) & \text{if } s(x) > 0 \end{cases}$$

   (a) Prove $f(w_n) = w_{n+1}$ for $w_n : \mathsf{State} \rightharpoonup \mathsf{State}$ as defined in the lecture, for our notion of State.

   (b) Prove $f(w_\infty) = w_\infty$ for $w_\infty : \mathsf{State} \rightharpoonup \mathsf{State}$ as defined in the lecture, for our notion of State.

   (c) Show implication (3) on page 19, that is: for all w,

$$w = f(w) \Rightarrow w_\infty \sqsubseteq w.$$

   (d) Prove that $\forall\, s \in \mathsf{State}\, \exists\, n[f^n(\bot)(s) = f^{n+1}(\bot)(s)]$.

4. [Extra exercise to possibly think about] Define a denotational semantics for the statement **for** $x := e_1$ **to** e_2 **do** P:

   (a) First with e_1, e_2 fixed numbers in $\mathbb{Z}$, say n and m.

   (b) Discuss some of the choices and problems with giving the general semantics, where e_1 and e_2 are arbitrary expressions.
   What semantics would you give to **for** $x := 1$ **to** $x + 1$ **do skip** ? And to **for** $x := 1$ **to** 3 **do** $x := x - 1$?