Semantics and Domain theory

Exercises 3

NB. A domain is a cpo with a $\perp$-element.

1. Prove that the set of partial functions from X to Y, $X \rightharpoonup Y$, forms a domain, with the definitions of ordering and lub given on slide 24 of the lecture notes (or the equivalent definitions given at the lecture).

2. Which of the following partial orders is a domain? In each case, choose a proper definition of 'lub'; prove your answer. You don't have to prove it is a partial order, but please check for yourself that it is! (In case you haven't done so.)

   (a) $(\mathcal{P}(\mathbb{N}), \subseteq)$, where $\mathcal{P}(\mathbb{N})$ is the powerset of $\mathbb{N}$ (the set of all subsets of $\mathbb{N}$) and $\subseteq$ is the usual subset ordering.

   (b) $(\mathcal{P}_{\text{fin}}(\mathbb{N}), \subseteq)$, where $\mathcal{P}_{\text{fin}}(\mathbb{N})$ is the set of finite subsets of $\mathbb{N}$ and $\subseteq$ is the usual subset ordering.

   (c) $([0,1], \leq)$, where $[0,1]$ is the unit interval and $\leq$ is the usual ordering on the real numbers.

   (d) $([0,1] \cap \mathbb{Q}, \leq)$, where $\cap$ is the intersection and $\mathbb{Q}$ is the set of rational numbers.

   (e) $(\Sigma^*, \sqsubseteq)$, where Σ^* is the set of words over the alphabet $\Sigma := \{a, b\}$ and $\sqsubseteq$ is the prefix ordering, defined by $w \sqsubseteq wv$ for all $w, v \in \Sigma^*$.

   (f) $(\Sigma^* \cup \Sigma^\omega, \sqsubseteq)$, where Σ^ω is the set of infinite words over the alphabet $\Sigma := \{a, b\}$ and $\sqsubseteq$ is the prefix ordering, defined by $w \sqsubseteq wv$ for all $w \in \Sigma^*$, $v \in \Sigma^* \cup \Sigma^\omega$ and $v \sqsubseteq v$ for all $v \in \Sigma^* \cup \Sigma^\omega$.

3. Prove that the function $f_{b,C}$ in the definition of the denotational semantics of **while** b **do** C (slide 11) is continuous. When is $f_{b,C}$ strict?

4. Let $(d_i)_{i\geq 0}$ and $(e_i)_{i\geq 0}$ be chains in a domain $(D, \sqsubseteq)$. Suppose that $(d_i)_{i\geq 0}$ is *majorized by* $(e_i)_{i\geq 0}$, that is: $\forall\, i\, \exists\, j (d_i \sqsubseteq e_j)$.
   Prove that $\sqcup_{i\geq 0} d_i \sqsubseteq \sqcup_{i\geq 0} e_i$.

5. Let $(E, \sqsubseteq')$ be a domain and suppose that in the domain $(D, \sqsubseteq)$, all chains are *eventually constant*, that is: for all chains $(d_i)_{i\geq 0}$ there exists an n such that $d_n = d_{n+1} = d_{n+2} = \ldots$
   Show that every monotonic $f : D \to E$ is continuous.

6. Prove that, given the partial function $f : X \rightharpoonup Y$, the function $f_\perp : X_\perp \to Y_\perp$ is continuous. (Proposition 3.1.1 in Pitts' notes.) Here $f_\perp$ is defined by

$$f_\perp(d) := \begin{cases} f(d) & \text{if } d \in X \text{ and } f(d)\downarrow \\ \perp & \text{if } d \in X \text{ and } f(d)\uparrow \\ \perp & \text{if } d = \perp \end{cases}$$

7. Consider the two-element domain $\mathbf{2} := \{\perp, \top\}$.

   (a) List all functions $h : \mathbf{2} \to \mathbf{2}$.

   (b) Which ones are monotonic and which ones are not? Draw a Hasse diagram depicting the ordering of the monotonic functions from $\mathbf{2}$ to $\mathbf{2}$.