Semantics and Domain theory

Exercises 6

1. (a) Show that if S and T are chain-closed, then $S \cup T$ is chain-closed.

   (b) Show that if S_i is chain-closed for every $i \in I$, then $\bigcap_{i \in I} S_i$ is chain-closed.

2. (Exercise 4.4.2. of Pitts' notes) Give an example of a subset S of $D \times D$ that is not chain-closed, but which satisfies:

$$\begin{array}{ll} (a) & \forall d \in D,\ \{e | (d, e) \in S\} \text{ is chain-closed} \\ (b) & \forall e \in D,\ \{d | (d, e) \in S\} \text{ is chain-closed.} \end{array}$$

   [Hint: consider $D = D = \Omega$, the cpo in Figure 1.] (Compare this with the property of continuous functions given on Slide 33 of Pitts' notes.)

3. The collection of chain-closed sets is not closed under arbitrary union. (It is not the case, in general, that $\forall i \in I (S_i$ is chain closed$)$ implies $\bigcup_{i \in I} S_i$ is chain-closed.)

   (a) Conclude this from the previous exercise.

   (b) Conclude this by directly constructing a counterexample in Ω.

4. Prove that for $f : D \to E$ monotonic,

$$f^{-1} \text{ preserves chain-closed sets} \Rightarrow f \text{ is continuous,}$$

   where f^{-1} *preserves chain-closed sets* means that, for all $S \subseteq E$, if S is chain-closed, then $f^{-1}(S)$ is a chain-closed subset of D.

5. Show that the untyped λ-term ω $(= \lambda x.x\,x)$ is not typable in PCF. That is: show that there are no τ_1 and τ_2 such that $\vdash \mathbf{fn}\ x : \tau_1.\,(x\ x) \ : \ \tau_2$.

6. (a) Suppose that the term mult : **nat** $\to$ **nat** $\to$ **nat** defines multiplication in PCF. Give a PCF term that defines the exponentiation function exp : **nat** $\to$ **nat** $\to$ **nat**. (So $\exp n\, m$ should denote n^m; you don't have to prove that exp correctly defines exponentiation.)

   (b) Let a PCF term p : **nat** $\to$ **nat** be given. Define a term N : **nat** that denotes the smallest number n such that $p(n) = 0$ and $\forall i < n(p(n) > 0)$. (You don't have to prove the correctness of N.)