

Semantics and Domain theory

Exercises 12

1. For a continuous function $f : \wp(D) \rightarrow \wp(D)$, we have defined the trace of f as $\text{Tr}(f) := \{(\alpha, a) \mid \alpha \text{ finite and } a \in f(\alpha)\}$. Prove the following properties (which also completes the proof of Lemma 2.20 of the notes).

- (a) $f(X) = \{a \mid \exists \alpha \subseteq_{\text{fin}} X ((\alpha, a) \in \text{Tr}(f))\}$.
- (b) We have $f \sqsubseteq g \iff \text{Tr}(f) \subseteq \text{Tr}(g)$, (for $g : \wp(D) \rightarrow \wp(D)$ continuous).
- (c) For $R \subseteq \wp(\wp_{\text{fin}}(D) \times D)$, we define $f_R : \wp(D) \rightarrow \wp(D)$ by

$$f_R(X) := \{a \mid \exists \alpha \subseteq_{\text{fin}} X ((\alpha, a) \in R)\}.$$

Prove that f_R is continuous.

2. At the lecture, we have seen the interpretations in the model \mathcal{E} of **I** ($= \lambda x.x$), **K** ($= \lambda x.\lambda y.x$) and **II**.

- (a) Compute the interpretation of $\lambda x.x x$. Is this the empty set?
- (b) Show that $\llbracket \mathbf{KI} \rrbracket = \{(\beta, (\gamma, c)) \mid c \in \gamma\}$ (without doing a β -reduction first).

3. Let d_0 be an element of $\wp(D_A)$ and let ρ be a valuation with $\rho(y) = d_0$.

- (a) Compute the interpretation in $\wp(D_A)$ of $\lambda x.y x$ by expressing $\llbracket \lambda x.y x \rrbracket_\rho$ in terms of d_0 .
- (b) Conclude that the η -rule does not hold in \mathcal{E} . (The η -rule says that $\lambda x.M x = M$ if $x \notin \text{FV}(M)$.)

4. Use the result of the following exercise ($\llbracket \Omega \rrbracket = \emptyset$) to:

- (a) Compute the interpretation of $\lambda y.\Omega$ in \mathcal{E} .
- (b) Compute the interpretation of $\lambda y.y \Omega$ in \mathcal{E} .

5. [Challenging] Show that the interpretation of $\Omega (= (\lambda x.x x)(\lambda x.x x))$ in \mathcal{E} is \emptyset .

(Hint: From a $c \in \llbracket \Omega \rrbracket$ you can construct an infinite sequence $(\alpha_i)_{i \in \mathbb{N}}$ with $(\alpha_{i+1}, c) \in \alpha_i \subseteq \alpha_0$ for all i , which is impossible in \mathcal{E} .)

6. Which of the following sets are complete lattices.

- (a) The set of flat natural numbers \mathbb{N}_\perp .
- (b) The set $\mathcal{P}_{\text{fin}}(\mathbb{N})$ of *finite subsets* of \mathbb{N} .
- (c) The set $\Omega (= \mathbb{N} \cup \{\omega\}$, with the ordering we have seen before).
- (d) The set of monotone functions from \mathbb{B}_\perp^\top to \mathbb{B}_\perp^\top .
(Remember that the set of flat booleans with a top element added, \mathbb{B}_\perp^\top , is a complete lattice.)

7. Complete the proof of Proposition 2.10.

That is, show that in a complete lattice (D, \sqsubseteq) , if we define

$$\bigsqcap X := \bigsqcup \{y \in D \mid y \sqsubseteq X\},$$

then $\bigsqcap X$ is indeed the *greatest lower bound* (also called the *glb*) of X .

8. Prove that in a complete lattice, the *absorption law* holds: $x \sqcup (x \sqcap y) = x$,