Semantics and Domain theory

Exercises 4

1. Let $(D, \sqsubseteq)$ be the domain of finite and infinite sequences over $\Sigma := \{a, b\}$ with $\sqsubseteq$ the prefix ordering. (So $D = \Sigma^* \cup \Sigma^\omega$.)

   (a) Which of the following functions $f : D \to D$ is monotonic / continuous?

   i. $f(s) = s$ with all a's removed.

   ii. $f(s) = abba$ if s is finite; $f(s) = s$ if s is infinite.

   iii. $f(s) = abbas$.

   iv. $f(s) = a$ if s contains finitely many b's; $f(s) = b$ if s contains infinitely many b's

   (b) For each of the functions f in (a) that is continuous, compute the least fixed point of f.

2. Let $(D, \sqsubseteq)$ be a domain with some element d_0 and let $f : D \to D$ be continuous. Suppose $d_0 \sqsubseteq f(d_0)$. Prove that $\sqcup_{i \in \mathbb{N}} f^i(d_0)$ is a fixed point of f.

3. Let $f, g : (D, \sqsubseteq) \to (D, \sqsubseteq)$ be continuous functions on domain $(D, \sqsubseteq)$. Prove

$$\mathrm{fix}(f \circ g) = f(\mathrm{fix}(g \circ f))$$

   (a) by unfolding the *definition* of fix (slide 29)

   (b) by using the *properties* of pre-fixed point (slide 20) and fixed point (slide 29) and proving

   i. $\mathrm{fix}(f \circ g) \sqsubseteq f(\mathrm{fix}(g \circ f))$

   ii. $f(\mathrm{fix}(g \circ f)) \sqsubseteq \mathrm{fix}(f \circ g)$

4. (Exercise 3.4.2 of Fiore's notes): Let X and Y be sets and $X_\perp$ and $Y_\perp$ be the corresponding flat domains. Show that a function $f : X_\perp \to Y_\perp$ is continuous if and only if one of (a) or (b) holds:

   (a) f is strict, i.e. $f(\perp) = \perp$.

   (b) f is constant, i.e. $\forall\, x \in X(f(x) = f(\perp))$.

5. For the *disjoint union* of two domains (also called the *binary sum* of domains), there are two choices: the *coalesced sum* (or *smashed sum*) $D +_c E$, or the *separated sum* $D +_s E$.

   For the coalesced sum, the set $D +_c E$ is defined as

$$\{\perp\} \cup \{(0, d) \mid d \in D, d \neq \perp_D\} \cup \{(1, e) \mid e \in E, e \neq \perp_E\}$$

   For the separated sum, the set $D +_s E$ is defined as

$$\{\perp\} \cup \{(0, d) \mid d \in D\} \cup \{(1, e) \mid e \in E\}$$

   So, the separated sum introduces a new $\perp$ element, whereas the coalesced sum "coalesces (or smashes) them together".

   (NB. The 0 and 1 in the pairs have no special significance, apart from being able to distinguish the "elements coming from D" from the "elements coming from E"; we want to define the *disjoint* union, which should also work, for example, for $\mathbb{N}_\perp + \mathbb{N}_\perp$.)

   Let two domains $(D, \sqsubseteq_D)$ and $(E, \sqsubseteq_E)$ be given.

(a) Define the partial ordering $\sqsubseteq$ on $D +_s E$ and give the $\perp$-element.

(b) Define the partial ordering $\sqsubseteq$ on $D +_c E$ and give the $\perp$-element.

(c) For $(f_i)_{i\in\mathbb{N}}$ a chain in $D +_s E$ define $\sqcup_{i\in\mathbb{N}} f_i$ and prove that it is the least upperbound.

(d) For $(f_i)_{i\in\mathbb{N}}$ a chain in $D +_c E$ define $\sqcup_{i\in\mathbb{N}} f_i$ and prove that it is the least upperbound.

(e) Define injections $\mathsf{inl} : D \to D +_s E$ and $\mathsf{inr} : E \to D +_s E$ that are continuous. (You don't have to prove that they are continuous.)

(f) Define injections $\mathsf{inl} : D \to D +_c E$ and $\mathsf{inr} : E \to D +_c E$ that are continuous. (You don't have to prove that they are continuous.)

(g) (*) For F a domain and $f : D \to F$, $g : E \to F$ we want to define a continuous function $[f, g] : D + E \to F$ such that

$$\begin{aligned} [f,g](\mathsf{inl}(x)) &= f(x), \\ [f,g](\mathsf{inr}(x)) &= g(x). \end{aligned}$$

Show how to define $[f, g]$ for the case of $D +_c E$ and for the case of $D +_s E$. For one of these cases, we can only define $[f, g]$ if we place additional requirements on f and g. Which?

6. (*) [If you are familiar with Topology, this is to show that "topological continuity" is the same as "Scott continuity", the notion of continuity that we use in Domain Theory]
For $(D, \sqsubseteq)$ a domain, we say that $X \subseteq D$ is *open* in case we have

(i) $\forall\, x, x' \in D(x \in X \wedge x \sqsubseteq x' \Rightarrow x' \in X)$ "X is an upperset".

(ii) for all chains $(d_i)_{i\geq 0}$, $\sqcup_{i\geq 0} d_i \in X \Rightarrow \exists\, i(d_i \in X)$ "X is inaccessible by chains".

For D and E domains, we say that $f : D \to E$ is *topologically continuous* if

$$\forall\, Y \subseteq E(Y \text{ is open } \Rightarrow f^{-1}(Y) \text{ is open}),$$

where $f^{-1}(Y) := \{x \in X \mid f(x) \in Y\}$.

(a) Prove that the set $\{x \in D \mid x \not\sqsubseteq d\}$ is open (for al $d \in D$).

(b) Prove that f is Scott-continuous implies f is topologically continuous.

(c) Prove that f is topologically continuous implies f is Scott-continuous. Hint: use contraposition.