Semantics and Domain theory

Exercises 5

1. Prove that for D a domain and $F : (D \to D) \to (D \to D)$ and $g : D \to D$ continuous,

$$\mathrm{ev}(\mathrm{fix}(F), \mathrm{fix}(g)) = \sqcup_{k \geq 0} F^k(\bot')(g^k(\bot)),$$

where $\bot$ is in D and $\bot'$ is in $D \to D$ and ev is the evaluation function (of Proposition 3.3.1):

$$\mathrm{ev}(f, d) := f(d),$$

for $f : D \to D$ and $d : D$, so $\mathrm{ev} : (D \to D) \times D \to D$.

2. We define two variants of a functional $F : [\mathbb{N}_\bot \to \mathbb{N}_\bot] \to \mathbb{B}_\bot$ that, given a continuous function $f : \mathbb{N}_\bot \to \mathbb{N}_\bot$, checks if f has a fixed point in $\mathbb{N}$. (That is: an $n \in \mathbb{N}$ for which $f(n) = n$).

$$F_1(f) \quad := \quad \begin{cases} \mathrm{tt} & \text{if } f \text{ is total and } \exists\, n \in \mathbb{N}(f(n) = n) \\ \mathrm{ff} & \text{if } f \text{ is total and } \forall\, n \in \mathbb{N}(f(n) \neq n) \\ \bot & \text{if } f \text{ is not total} \end{cases}$$

$$F_2(f) \quad := \quad \begin{cases} \mathrm{tt} & \text{if } \exists\, n \in \mathbb{N}(f(n) = n \wedge \forall\, m < n(f(m) \neq m, \bot)) \\ \bot & \text{otherwise} \end{cases}$$

NB. "f is total" means that $\forall\, n \in \mathbb{N}(f(n) \neq \bot)$.

   (a) Prove that both F_1 and F_2 are monotonic.

   (b) Prove that one of the F_i does not preserve lubs (and thus is not continuous).

   (c) Prove that one of the F_i preserves lubs (and thus is continuous).

3. Let $P : D \to \mathbb{B}_\bot$ and $g : D \to D$ be continuous. Define $f : D \times D \to D \times D$ by

$$f(d_1, d_2) = \mathrm{If}(P(d_1), (g(d_1), g(d_2)), (g(d_2), g(d_1))).$$

Show that for $\mathrm{fix}(f) = (u_1, u_2)$, we have $u_1 = u_2$. (Use Scott induction.)

4. Show that the following two definitions of the ordering between continuous functions $f, g : D \to E$ (see Slide 35) are equivalent.

   (a) $f \sqsubseteq g := \forall\, d \in D(f(d) \sqsubseteq_E g(d))$.

   (b) $f \sqsubseteq' g := \forall\, d_1, d_2 \in D(d_1 \sqsubseteq_D d_2 \Rightarrow f(d_1) \sqsubseteq_E g(d_2))$.

5. [Borrowed from Fiore] We consider the two-element domain $\mathbf{2} := \{\bot, \top\}$. For $(D, \sqsubseteq)$ a domain and $d \in D$, define $g_d : D \to \mathbf{2}$ by

$$g_d(x) := \begin{cases} \bot & \text{if } x \sqsubseteq d \\ \top & \text{if } x \not\sqsubseteq d \end{cases}$$

   (a) Prove that g_d is continuous (for any $d \in D$).

   (b) Let E be some domain. Prove that $f : E \to D$ is continuous iff $g_d \circ f$ is continuous for all $d \in D$. (We know that the composition of two continuous functions is continuous.)