

Semantics and Domain theory

Exercises 6

1. (a) Show that if S and T are chain-closed, then $S \cup T$ is chain-closed.
 (b) Show that if S_i is chain-closed for every $i \in I$, then $\bigcap_{i \in I} S_i$ is chain-closed.
2. (Exercise 4.4.2. of Fiore's notes) Give an example of a subset S of $D \times D$ that is not chain-closed, but which satisfies:

- (a) $\forall d \in D, \{e \mid (d, e) \in S\}$ is chain-closed
- (b) $\forall e \in D, \{d \mid (d, e) \in S\}$ is chain-closed.

[Hint: consider $D = D = \Omega$, the cpo in Figure 1.] (Compare this with the property of continuous functions given on Slide 33 of Fiore's notes.)

3. The collection of chain-closed sets is not closed under arbitrary union. (It is not the case, in general, that $\forall i \in I (S_i \text{ is chain closed})$ implies $\bigcup_{i \in I} S_i$ is chain-closed.)
 (a) Conclude this from the previous exercise.
 (b) Conclude this by directly constructing a counterexample in Ω .
4. Prove that for $f : D \rightarrow E$ monotonic,

$$f^{-1} \text{ preserves chain-closed sets } \Rightarrow f \text{ is continuous,}$$

where f^{-1} *preserves chain-closed sets* means that, for all $S \subseteq E$, if S is chain-closed, then $f^{-1}(S)$ is a chain-closed subset of D .

5. Show that the untyped λ -term $\omega (= \lambda x. x x)$ is not typable in PCF. That is: show that there are no τ_1 and τ_2 such that $\vdash \mathbf{fn} \ x : \tau_1. (x \ x) \quad : \quad \tau_2$.
6. (a) Suppose that the term $\mathbf{mult} : \mathbf{nat} \rightarrow \mathbf{nat} \rightarrow \mathbf{nat}$ defines multiplication in PCF. Give a PCF term that defines the exponentiation function $\mathbf{exp} : \mathbf{nat} \rightarrow \mathbf{nat} \rightarrow \mathbf{nat}$. (So $\mathbf{exp} \ n \ m$ should denote n^m ; you don't have to prove that $\mathbf{exp}$ correctly defines exponentiation.)
 (b) Let a PCF term $p : \mathbf{nat} \rightarrow \mathbf{nat}$ be given. Define a term $N : \mathbf{nat}$ that denotes the smallest number n such that $p(n) = 0$ and $\forall i < n (p(i) > 0)$. (You don't have to prove the correctness of N .)