

Semantics and Domain theory

Exercises 8

1. (Exercise 6.5.2. of Fiore's notes) Define $\Omega_\tau = \mathbf{fix}(\mathbf{fn} x : \tau.x)$
 - (a) Show that $\llbracket \Omega_\tau \rrbracket$ is the least element of the domain $\llbracket \tau \rrbracket$.
 - (b) Deduce that $\llbracket \mathbf{fn} x : \tau.\Omega_\tau \rrbracket = \llbracket \Omega_{\tau \rightarrow \tau} \rrbracket$.
2. (a) Compute the denotational semantics of $M = \mathbf{fn} x : \mathbf{bool}.\mathbf{fn} y : \mathbf{nat}.\mathbf{if} x \mathbf{then} y \mathbf{else} y$.
 - (b) Define a term P such that $\llbracket M \rrbracket \subseteq \llbracket P \rrbracket$ but $\llbracket M \rrbracket \neq \llbracket P \rrbracket$.
 - (c) Define a term N such that $\llbracket M \rrbracket = \llbracket N \rrbracket$ but $N \not\Downarrow M$.
3. Define terms $M, N : \mathbf{nat} \rightarrow \mathbf{nat}$ with $\llbracket M \rrbracket \subseteq \llbracket N \rrbracket$ and $\llbracket M \rrbracket \neq \llbracket N \rrbracket$.
4. Verify that $\llbracket (\mathbf{fn} x : \sigma.M)N \rrbracket(\rho) = \llbracket M[N/x] \rrbracket(\rho)$ for M, N with $\Gamma \vdash N : \sigma$ and $\Gamma, x : \sigma \vdash M : \tau$ and $\rho \in \llbracket \Gamma \rrbracket$. (Use the result on Slide 62, the Substitution Lemma.) ;
5. Prove Soundness of the operational semantics (Theorem 6.4.1) for the inductive cases \Downarrow_{pred} and \Downarrow_{if1} .
Remember that Theorem 6.4.1 states that for all closed expressions M and V and type τ , if $M \Downarrow_\tau V$, then $\llbracket M \rrbracket = \llbracket V \rrbracket$. It is proved by induction on the derivation of $M \Downarrow_\tau V$.
6. Let

$$P := \mathbf{fix}(\mathbf{fn} p : \mathbf{nat} \rightarrow \mathbf{bool}.\mathbf{fn} x : \mathbf{nat}.\mathbf{if} (\mathbf{zero} x) \mathbf{then} \mathbf{true} \mathbf{else} p(\mathbf{pred}(\mathbf{pred} x))).$$

Compute $\llbracket P \rrbracket$.

7. We give PCF-terms that define the “or” function, taking partiality into account. A PCF-term M defines the domain-theoretic function f if $\llbracket M \rrbracket = f$.
 - (a) Give a PCF-term that defines the function $f : \mathbb{B}_\perp \rightarrow \mathbb{B}_\perp \rightarrow \mathbb{B}_\perp$ given by

$$f(x)(y) := \begin{cases} \perp & \text{if } x = \perp \text{ or } (x = \mathbf{ff} \text{ and } y = \perp) \\ \mathbf{tt} & \text{if } x = \mathbf{tt} \text{ or } (x = \mathbf{ff} \text{ and } y = \mathbf{tt}) \\ \mathbf{ff} & \text{if } x = \mathbf{ff} \text{ and } y = \mathbf{ff} \end{cases}$$

- (b) Give a PCF-terms that defines the function $g : \mathbb{B}_\perp \rightarrow \mathbb{B}_\perp \rightarrow \mathbb{B}_\perp$ given by

$$g(x)(y) := \begin{cases} \perp & \text{if } x = \perp \text{ or } y = \perp \\ \mathbf{tt} & \text{if } x, y \neq \perp \text{ and } (x = \mathbf{tt} \text{ or } y = \mathbf{tt}) \\ \mathbf{ff} & \text{if } x = \mathbf{ff} \text{ and } y = \mathbf{ff} \end{cases}$$