Herbrand Models and SLD resolution

Additional Course Notes for Semantcs of Logic Programming

Herman Geuvers

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1 Herbrand Models and Completeness of Logic Programming

1.1 Soundness and Completeness

In these notes we prove that the resolution method is *complete* for logic programs. So, if S is a finite set of program clauses and $\leftarrow A$ is a goal, then

$$\begin{array}{c}\text{There is a derivation of the empty clause from } S \cup \{\leftarrow A\} \text{ by resolution} \\ \Updownarrow \\ S \vdash A\end{array}$$

We have already proven soundness of the general resolution method, which means that we already have the implication $\Downarrow$. The implication $\Uparrow$ states completeness: if A is derivable from S, then there is a resolution derivation of $S \cup \{\leftarrow A\}$. (That is, a *refutation* of $\neg A$ from S.)

The proof of $\Uparrow$ proceeds as follows.

1. If $S \vdash A$, then $S \models A$.
   This is the *soundness* of predicate logic: if something is derivable, it is true in all models.

2. If S is a set of program clauses, then S has a *smallest Herbrand model* $\mathcal{M}_S$.
   The notion of *Herbrand model* is described in the rest of these notes.

3. If the goal A holds in the smallest Herbrand model of S, then $S \cup \{\leftarrow A\}$ has an SLD refutation.
   The definition of SLD resolution can be found in [vanBenthem]. We do not repeat it here.

4. So, if $S \vdash A$, then $S \models A$, then $\mathcal{M}_S \models A$ (where $\mathcal{M}_S$ is the smallest Herbrand model of S), then $S \cup \{\leftarrow A\}$ has an SLD refutation, then (a fortiori) it has a resolution refutation, so we are done.

NB SLD is an abbreviation of "Selected-literal Linear resolution for Definite Clauses". ("Definite clause" is another word for program clause.)

1.2 Herbrand Models

Definition 1.1 *Given a finite set of formulas P in the language L, the* Herbrand universe *of L, H_L (also H_P) is the set of all closed terms that can be built up from symbols of the language L.*

The closed terms of a language are also called the *gound terms*. In case there is no constant, we just add one, to make sure that the universe is not empty.

Example 1.2 *Given $P :=$*

$$\begin{array}{rcl} path(x,z) & \leftarrow & arc(x,y), path(y,z) \\ path(x,x) & \leftarrow & \\ arc(a,b) & \leftarrow & \end{array}$$

we find that $H_P = \{a, b\}$.

Definition 1.3 *Given a language L, a* Herbrand interpretation of L *is a pair $\langle \mathcal{D}, \mathcal{I} \rangle$ with $\mathcal{D} = (D, R, O)$ such that*

1. *$D = H_L$,*
2. *$\mathcal{I}(c) = c$ for all constants c,*
3. *$\mathcal{I}(f)(t_1, \ldots, t_n) = f(\mathcal{I}(t_1), \ldots, \mathcal{I}(t_n))$,*
4. *$\mathcal{I}(P) \subseteq H_L^n$ if P is a predicate of arity n.*

So, for a Herbrand interpretation of L, D and O are fixed, but R can still be chosen. Often we will not describe R explicitly but just define $\mathcal{I}(P)$ for all predicate symbols P. In a Herbrand model, the ground terms have a fixed interpretation, as themselves.

Definition 1.4 *A* Herbrand model *for the set of formulas S (with language L) is a Herbrand interpretation in which all formulas of S are true.*

Example 1.5 *Look at the "Som" example of [vanBenthem].*

1. *$D := \mathbb{N}$, $\mathcal{I}(Som)(n, m, p) := p = n + m$. This is a Herbrand model of "Som".*
2. *$D := \mathbb{N}$, $\mathcal{I}(Som)(n, m, p) := true$. This is also a Herbrand model of "Som".*
3. *$D := \mathbb{Q}$, $\mathcal{I}(0) := 1$, $\mathcal{I}(S)(x) := 5x$, $\mathcal{I}(Som)(x, y, z) := x \cdot y = z$. This is a model of "Som", but not a Herbrand model of it.*

Example 1.6 *See Example 1.2*

1. $D := \{a, b\}$, $\mathcal{I}(arc)(x, y) := x = a \wedge y = b$, $\mathcal{I}(path)(x, y) := x = y \vee (x = a \wedge y = b)$. *This is a Herbrand model of P in 1.2. One can visualize this Herbrand model as a graph with two types of edges: "arcs" and "paths".*

2. *Another model of P of 1.2 is* $D := \{a, b, c, d, e, f, g, h\}$, $\mathcal{I}(arc)(x, y) := x < y$ *(lexicographically),* $\mathcal{I}(path)(x, y) := x \leq y$ *(lexicographically). This is not a Herbrand model.*

Recall that a *Horn clause* has exactly one positive literal. (So a goal is not a Horn cluase.)

Theorem 1.7 *If a set of Horn clauses S has a model, then S has a Herbrand model.*

Proof Let S be a set of Horn clauses over the language L and suppose that $\mathcal{M}$ is a model of S, say $\mathcal{M} = (\mathcal{D}, \mathcal{I})$. Define the Herbrand model $\mathcal{M}'$ of S as follows.

$\mathcal{M}' := (H_L, \mathcal{I}')$ with H_L the Herbrand universe of L and

$$\mathcal{I}'(P)(t_1, \ldots, t_n) \text{ iff } \mathcal{I}(P)(t_1, \ldots, t_n) \text{ holds in the model } \mathcal{M}.$$

(Actually we should write $\mathcal{I}(P)([\![t_1]\!], \ldots, [\![t_n]\!])$.) Then $\mathcal{M}'$ is a model of S: Suppose $C := B_1 \wedge \ldots \wedge B_n \rightarrow A \in S$, with $\mathrm{FV}(C) = \{x_1, \ldots, x_k\}$ and $\mathcal{M} \models C$. Then:
for all ground terms $t_1, \ldots, t_k$, $\mathcal{M} \models C[\vec{t}/\vec{x}]$
$\Longleftrightarrow$
for all ground terms $t_1, \ldots, t_k$, if $\mathcal{M} \models B_1[\vec{t}/\vec{x}], \ldots, \mathcal{M} \models B_n[\vec{t}/\vec{x}]$, then $\mathcal{M} \models A[\vec{t}/\vec{x}]$
$\stackrel{*}{\Longleftrightarrow}$
for all ground terms $t_1, \ldots, t_k$, if $\mathcal{M}' \models B_1[\vec{t}/\vec{x}], \ldots, \mathcal{M}' \models B_n[\vec{t}/\vec{x}]$, then $\mathcal{M}' \models A[\vec{t}/\vec{x}]$
$\Longleftrightarrow$
$\mathcal{M} \models C[\vec{t}/\vec{x}]$.
The $\stackrel{*}{\Longleftrightarrow}$ is because $\mathcal{M} \models A \Longleftrightarrow \mathcal{M}' \models A$ for closed atomic formulas A.

Example 1.8

1. $S := \{P(a), \exists x \neg P(x)\}$ *does not have a Herbrand model, but it does have a model. (Exercise: Construct a model of S and verify that S doesn't have a Herbrand model.)*

2. $S := \{Q(a) \vee P(a)\}$ *has a Herbrand model but not a smallest Herbrand model. (Exercise: Construct two Herbrand models of S.)*

Theorem 1.9 *If* $\mathcal{M}_1$ *and* $\mathcal{M}_2$ *are Herbrand models of the set of Horn clauses S, then* $\mathcal{M}_1 \cap \mathcal{M}_2$ *is also a Herbrand model of S.*

Proof $\mathcal{M} := \mathcal{M}_1 \cap \mathcal{M}_2$ is the model $(\mathcal{D}, \mathcal{I})$, where $\mathcal{I}$ is the interpretation $\mathcal{I}(P)(t_1, \ldots, t_n) := \mathcal{I}_1(P)(t_1, \ldots, t_n) \wedge \mathcal{I}_2(P)(t_1, \ldots, t_n)$.
For $C := B_1 \wedge \ldots \wedge B_n \to A \in S$, we have to prove that $\mathcal{M} \models C$. Suppose that $\mathrm{FV}(C) = \{x_1, \ldots, x_k\}$.
Let $t_1, \ldots, t_k \in D$ and suppose $\mathcal{M} \models B_1[\vec{t}/\vec{x}], \ldots, \mathcal{M} \models B_n[\vec{t}/\vec{x}]$. Then $\mathcal{M}_1 \models B_1[\vec{t}/\vec{x}], \ldots, \mathcal{M}_1 \models B_n[\vec{t}/\vec{x}]$ and $\mathcal{M}_2 \models B_1[\vec{t}/\vec{x}], \ldots, \mathcal{M}_2 \models B_n[\vec{t}/\vec{x}]$ (dy definition of $\cap$) and therefore $\mathcal{M}_1 \models A[\vec{t}/\vec{x}]$ and $\mathcal{M}_2 \models A[\vec{t}/\vec{x}]$. So, $\mathcal{M} \models A[\vec{t}/\vec{x}]$, which was to be proven.

To describe semantically what SLD resolution does, we are interested in the *smallest* Herbrand model of a set of Horn clauses. Here *smallest* is not to be understood in terms of the size of the domain, because that is fixed anyway.

Definition 1.10 *Consider two models $\mathcal{M}$ and $\mathcal{M}'$ for a language L. We say that $\mathcal{M}$ is* smaller *than $\mathcal{M}'$ if for all formulas φ in the language L, $\mathcal{M} \models \varphi \Rightarrow \mathcal{M}' \models \varphi$.*

So a model is smaller if it makes less formulas true. Often two models are incomparable (like the two models you've constructed in the second example of 1.8). Herbrand models of (sets of) Horn clauses have a nice property.

Corollary 1.11 *Every set of Horn clauses S has a smallest Herbrand model.*

The smallest Herbrand model of S can be obtained by taking

$$\mathcal{M}_S := \bigcap \{\mathcal{M} \mid \mathcal{M} \text{ is a Herbrand model of } S\},$$

the intersection of all Herbrand models of S. Another way is to define the smallest model 'incrementally', by starting from the structure H_L where only the facts (as given in S) are true and then adding information about H_L if it can be derived by one of the program clauses until no new information can be obtained.

Theorem 1.12 *Given a set of Horn clauses S and a goal $\leftarrow A$, the following are equivalent.*

1. *$S \cup \{\leftarrow A\}$ has an SLD refutation*
2. *$S \models A$*
3. *A holds in the smallest Herbrand model of S.*

Proof (2) $\Rightarrow$ (3) has already been proven.
(1) $\Rightarrow$ (2) has also been proven: If $S \cup \{\leftarrow A\}$ has an SLD refutation then $S \vdash A$ (by Soundness of resolution), so $S \models A$ (by Soundness of natural deduction).
(3) $\Rightarrow$ (1) we don't prove.