



Proof-term reductions for general forms of natural deduction

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Classical and Constructive Logic

Classically, the meaning of a propositional connective is fixed by its **truth table**. This immediately implies

- consistency,
- a decision procedure,
- completeness (w.r.t. Boolean algebra's).

Constructively the meaning of a connective is fixed by explaining what a **proof** is that involves the connective.

Basically, this explains the **introduction rule(s)** for each connective, from which the elimination rules follow (Prawitz)

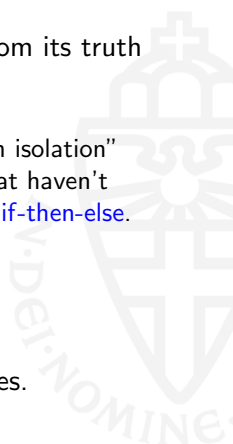
By analysing constructive proofs we then also get

- consistency (from proof normalization),
- a decision procedure (from the subformula property),
- Curry-Howard **proofs-as-terms** (and propositions-as-types) allowing to study normalization as term-reduction



This talk

- Derive natural deduction rules for a connective from its truth table definition.
 - For constructive logic!
 - Gives natural deduction rules for a connective “in isolation”
 - Also gives (constructive) rules for connectives that haven’t been studied constructively so far, like **nand** and **if-then-else**.
- Give proof-terms for natural deduction
- Study proof-normalization via term-reduction
- We define a general notion of detour conversion (cut-elimination) for these constructive connectives.





Standard form for natural deduction rules

$$\frac{\Gamma \vdash A_1 \quad \dots \quad \Gamma \vdash A_n \quad \Gamma, B_1 \vdash D \quad \dots \quad \Gamma, B_m \vdash D}{\Gamma \vdash D}$$

If the conclusion of a rule is $\Gamma \vdash D$, then the hypotheses of the rule can be of one of two forms:

- ① $\Gamma, B \vdash D$: we are given extra data B to prove D from Γ . We call B a Case.
- ② $\Gamma \vdash A$: instead of proving D from Γ , we now need to prove A from Γ . We call A a Lemma.

One obvious advantage: we don't have to give the Γ explicitly, as it can be retrieved:

$$\frac{\vdash A_1 \quad \dots \quad \vdash A_n \quad B_1 \vdash D \quad \dots \quad B_m \vdash D}{\vdash D}$$



Some well-known constructive rules

Rules that follow this format:

$$\frac{\vdash A \vee B \quad A \vdash D \quad B \vdash D}{\vdash D} \vee\text{-el} \qquad \frac{\vdash A \wedge B \quad A \vdash D}{\vdash D} \wedge\text{-el}$$

$$\frac{\vdash A \quad \vdash B}{\vdash A \wedge B} \wedge\text{-in}$$

Rule that does not follow this format:

$$\frac{A \vdash B}{\vdash A \rightarrow B} \rightarrow\text{-in}$$





Constructive natural deduction rules from truth tables

Let c be an n -ary connective c with truth table t_c .
 Each row of t_c gives rise to an elimination rule or an introduction rule for c . (We write $\varphi = c(A_1, \dots, A_n)$.)

elimination

$$\frac{\begin{array}{c|c} A_1 & \dots & A_n & \varphi \\ \hline p_1 & \dots & p_n & 0 \end{array}}{\vdash \varphi \dots \vdash A_j \text{ (if } p_j = 1) \dots A_i \vdash D \text{ (if } p_i = 0) \dots} \vdash D \quad \text{el}$$

introduction

$$\frac{\begin{array}{c|c} A_1 & \dots & A_n & \varphi \\ \hline q_1 & \dots & q_n & 1 \end{array}}{\dots \vdash A_j \text{ (if } q_j = 1) \dots A_i \vdash \varphi \text{ (if } q_i = 0) \dots} \vdash \varphi \quad \text{in}^i$$

This is the **constructive** introduction rule; there is also a **classical** introduction rule, which we don't discuss now.



Definition of the logics

Given a set of connectives $\mathcal{C} := \{c_1, \dots, c_n\}$, the **constructive** natural deduction systems for \mathcal{C} , $\text{IPC}_{\mathcal{C}}$, has the following rules.

- The **axiom rule**

$$\frac{}{\Gamma \vdash A} \text{ axiom} \quad (\text{if } A \in \Gamma)$$

- The **constructive introduction rules** for the connectives in \mathcal{C} , as derived from the truth table.
- The **elimination rules** for the connectives in \mathcal{C} , as derived from the truth table.



Examples

Derivation rules for \wedge (3 elim rules and one intro rule):

A	B	$A \wedge B$
0	0	0
0	1	0
1	0	0
1	1	1

$$\frac{\vdash A \wedge B \quad A \vdash D \quad B \vdash D}{\vdash D} \wedge\text{-el}_a$$

$$\frac{\vdash A \wedge B \quad A \vdash D \quad \vdash B}{\vdash D} \wedge\text{-el}_b$$

$$\frac{\vdash A \wedge B \quad \vdash A \quad B \vdash D}{\vdash D} \wedge\text{-el}_c$$

$$\frac{\vdash A \quad \vdash B}{\vdash A \wedge B} \wedge\text{-in}$$

- These rules can be shown to be equivalent to the well-known derivation rules.
- These rules can be optimized to 3 rules.



Examples

Rules for \neg : 1 elimination rule and 1 introduction rule.

A	$\neg A$
0	1
1	0

Derivation rules:

$$\frac{\vdash \neg A \quad \vdash A}{\vdash D} \neg\text{-el} \qquad \frac{A \vdash \neg A}{\vdash \neg A} \neg\text{-in}$$





Lemma 1 to simplify the rules

$$\frac{\vdash A_1 \dots \vdash A_n \quad B_1 \vdash D \dots B_m \vdash D \quad C \vdash D}{\vdash D}$$

$$\frac{\vdash A_1 \dots \vdash A_n \quad \vdash C \quad B_1 \vdash D \dots B_m \vdash D}{\vdash D}$$

is equivalent to the system with these two rules replaced by

$$\frac{\vdash A_1 \dots \vdash A_n \quad B_1 \vdash D \dots B_m \vdash D}{\vdash D}$$



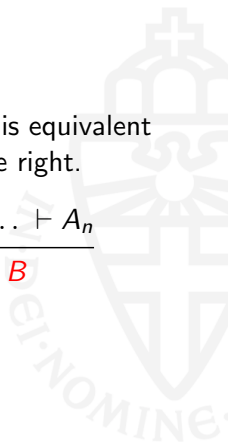


Lemma II to simplify the rules

A system with a deduction rule of the form to the left is equivalent to the system with this rule replaced by the rule on the right.

$$\frac{\vdash A_1 \dots \vdash A_n \quad B \vdash D}{\vdash D}$$

$$\frac{\vdash A_1 \dots \vdash A_n}{\vdash B}$$





The well-known constructive connectives

We have already seen the \wedge, \neg rules. The optimised rules for \vee, \rightarrow, \top and \perp we obtain are:

$$\frac{\vdash A \vee B \quad A \vdash D \quad B \vdash D}{\vdash D} \vee\text{-el} \qquad \frac{\vdash A}{\vdash A \vee B} \vee\text{-in}_1 \qquad \frac{\vdash B}{\vdash A \vee B} \vee\text{-in}_2$$

$$\frac{\vdash A \rightarrow B \quad \vdash A}{\vdash B} \rightarrow\text{-el} \qquad \frac{\vdash B}{\vdash A \rightarrow B} \rightarrow\text{-in}_1 \qquad \frac{A \vdash A \rightarrow B}{\vdash A \rightarrow B} \rightarrow\text{-in}_2$$

$$\frac{}{\vdash \top} \top\text{-in} \qquad \frac{\vdash \perp}{\vdash D} \perp\text{-el}$$



Substituting a deduction in another

LEMMA: If $\Delta, \varphi \vdash \psi$ and $\Gamma \vdash \varphi$, then $\Delta, \Gamma \vdash \psi$

If Π is a deduction of $\Delta, \varphi \vdash \psi$ and Σ is a deduction of $\Gamma \vdash \varphi$, then we have the following deduction of $\Delta, \Gamma \vdash \psi$:

$$\begin{array}{c}
 \boxed{\Sigma} \quad \dots \quad \boxed{\Sigma} \\
 \Gamma \vdash \varphi \quad \dots \quad \Gamma \vdash \varphi \\
 \boxed{\Pi} \\
 \Delta, \Gamma \vdash \psi
 \end{array}$$

In Π , every application of an (axiom) rule at a leaf, deriving $\Delta' \vdash \varphi$ for some $\Delta' \supseteq \Delta$ is replaced by a copy of a deduction Σ , which is also a deduction of $\Delta', \Gamma \vdash \varphi$.



Detours (cuts) in constructive logic

Remember that the rules for c arise from rows in the truth table t_c :

A_1	...	A_n	$c(A_1, \dots, A_n)$
p_1	...	p_n	0
q_1	...	q_n	1

DEFINITION A **detour convertibility** is a pattern of the following form, where $\varphi = c(A_1, \dots, A_n)$.

$$\frac{\dots \frac{\boxed{\Sigma_j}}{\Gamma \vdash A_j} \dots \frac{\boxed{\Sigma_i}}{\Gamma, A_i \vdash \varphi} \dots}{\Gamma \vdash \varphi} \text{ in} \quad \frac{\dots \frac{\boxed{\Pi_k}}{\Gamma \vdash A_k} \dots \frac{\boxed{\Pi_\ell}}{\Gamma, A_\ell \vdash D} \dots}{\Gamma \vdash D} \text{ el}$$

- $q_j = 1$ for A_j and $q_i = 0$ for A_i
- $p_k = 1$ for A_k and $p_\ell = 0$ for A_ℓ



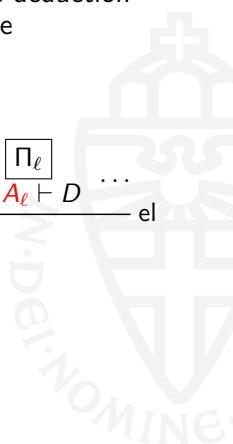
Eliminating a detour (detour conversion) (I)

The *elimination of a detour* is defined by replacing the deduction pattern by another one. If $j = \ell$ (for some j, ℓ), replace

$$\frac{\begin{array}{c} \dots \frac{\boxed{\Sigma_j}}{\Gamma \vdash A_j} \dots \frac{\boxed{\Sigma_i}}{\Gamma, A_i \vdash \varphi} \dots \\ \Gamma \vdash \varphi \end{array} \text{ in} \quad \begin{array}{c} \dots \frac{\boxed{\Pi_k}}{\Gamma \vdash A_k} \dots \frac{\boxed{\Pi_\ell}}{\Gamma, A_\ell \vdash D} \dots \\ \Gamma \vdash D \end{array} \text{ el}}{\Gamma \vdash D}$$

by

$$\begin{array}{c} \frac{\boxed{\Sigma_j}}{\Gamma \vdash A_j} \dots \frac{\boxed{\Sigma_j}}{\Gamma \vdash A_j} \\ \frac{\boxed{\Pi_\ell}}{\Gamma \vdash D} \end{array}$$





Eliminating a detour (detour conversion) (II)

If $i = k$ (for some i, k), replace

$$\frac{\dots \frac{\boxed{\Sigma_j}}{\Gamma \vdash A_j} \dots \frac{\boxed{\Sigma_i}}{\Gamma, A_i \vdash \varphi} \dots}{\Gamma \vdash \varphi} \text{ in } \dots \frac{\boxed{\Pi_k}}{\Gamma \vdash A_k} \dots \frac{\boxed{\Pi_\ell}}{\Gamma, A_\ell \vdash D} \dots}{\Gamma \vdash D} \text{ el}$$

by

$$\frac{\frac{\frac{\boxed{\Pi_k}}{\Gamma \vdash A_k} \dots \frac{\boxed{\Pi_k}}{\Gamma \vdash A_k}}{\boxed{\Sigma_i}}{\Gamma \vdash \varphi} \dots \frac{\boxed{\Pi_k}}{\Gamma \vdash A_k} \dots \frac{\boxed{\Pi_\ell}}{\Gamma, A_\ell \vdash D} \dots}{\Gamma \vdash D} \text{ el}$$



Observation

$$\frac{
 \begin{array}{c}
 \dots \quad \boxed{\Sigma_j} \quad \Gamma \vdash A_j \quad \dots \quad \boxed{\Sigma_i} \quad \Gamma, A_i \vdash \varphi \quad \dots \\
 \hline
 \Gamma \vdash \varphi
 \end{array}
 \text{ in }
 \begin{array}{c}
 \dots \quad \boxed{\Pi_k} \quad \Gamma \vdash A_k \quad \dots \quad \boxed{\Pi_\ell} \quad \Gamma, A_\ell \vdash D \quad \dots \\
 \hline
 \Gamma \vdash D
 \end{array}
 }{
 \Gamma \vdash D
 }
 \text{ el}$$

- There can be several “matching” (i, k) or (j, ℓ) pairs.
- So: cut-elimination is non-deterministic in general.



Permutation convertibility: Example

$$\frac{\frac{\frac{\frac{\vdash A \vee B}{\vdash A \vee B} \quad \frac{A, C \vdash C \rightarrow D}{A \vdash C \rightarrow D} \rightarrow -in_a}{\vdash C \rightarrow D} \vee -el \quad B \vdash C \rightarrow D}{\vdash C \rightarrow D} \vee -el \quad \frac{\vdash C \rightarrow D \quad \vdash C}{\vdash D} \rightarrow -el}{\vdash D} \rightarrow -el$$

A detour convertibility arising from $\rightarrow -in_a$ followed by $\rightarrow -el$ is blocked by the $\vee -el$. This is a **permutation convertibility**, which can be contracted by permuting the $\rightarrow -el$ rule over the $\vee -el$ rule.



Permutation convertibility: Definition

Let c and c' be connectives of arity n and n' , with elimination rules r and r' respectively. A **permutation convertibility** in a derivation is a pattern of the following form, where $\Phi = c(B_1, \dots, B_n)$, $\Psi = c'(A_1, \dots, A_{n'})$.

$$\frac{\frac{\vdots \boxed{\Sigma_j} \quad \vdots \boxed{\Sigma_i}}{\vdash \Psi \dots \vdash A_j \quad \dots \quad A_i \vdash \Phi \quad \dots} \text{el}_{r'}}{\vdash \Phi} \quad \frac{\vdots \boxed{\Pi_k} \quad \vdots \boxed{\Pi_\ell}}{\dots \vdash B_k \quad \dots \quad B_\ell \vdash D \quad \dots} \text{el}_r}{\vdash D}$$

- A_j ranges over all propositions that have a 1 in the truth table of c' ; A_i ranges over all propositions that have a 0,
- B_k ranges over all propositions that have a 1 in the truth table of c ; B_ℓ ranges over all propositions that have a 0.



Permutation conversion

The **permutation conversion** is defined by replacing the derivation pattern on the previous slide by

$$\frac{\vdash \Psi \dots \vdash A_j \quad \dots \quad \frac{\vdots \boxed{\Sigma_j} \quad \vdots \boxed{\Sigma_i} \quad A_i \vdash \Phi \quad \dots \quad \vdots \boxed{\Pi_k} \quad A_i \vdash B_k \quad \dots \quad \vdots \boxed{\Pi_\ell} \quad A_i, B_\ell \vdash D \quad \dots}{A_i \vdash D} \text{el}_r}{\vdash D} \text{el}_{r'}$$

This gives rise to copying of sub-derivations: for every A_i we copy the sub-derivations Π_1, \dots, Π_n .



Permutation conversion: Example

A permutation conversion replaces the following derivation

$$\frac{\frac{\frac{\vdash A \vee B}{\vdash A \vee B} \quad \frac{A, C \vdash C \rightarrow D}{A \vdash C \rightarrow D} \rightarrow -in_a \quad B \vdash C \rightarrow D}{\vdash C \rightarrow D} \text{V-el} \quad \frac{\vdash C \rightarrow D \quad \vdash C}{\vdash D} \rightarrow -el$$

by

$$\frac{\frac{\frac{\vdash A \vee B}{\vdash A \vee B} \quad \frac{A, C \vdash C \rightarrow D}{A \vdash C \rightarrow D} \rightarrow -in_a \quad A \vdash C}{A \vdash D} \rightarrow -el \quad \frac{B \vdash C \rightarrow D \quad B \vdash C}{B \vdash D} \rightarrow -el}{\vdash D} \text{V-el}$$



Curry-Howard proofs-as-terms

We define rules for the judgment $\Gamma \vdash t : A$, where

- A is a formula,
- Γ is a set of declarations $\{x_1 : A_1, \dots, x_m : A_m\}$, where the A_i are formulas and the x_i are term-variables,
- t is a **proof-term**:

$$t ::= x \mid \{\bar{t}; \overline{\lambda x : A.t}\}_r \mid t \cdot_r [\bar{t}; \overline{\lambda x : A.t}]$$

where x ranges over variables and r ranges over the rules.

For a connective $c \in \mathcal{C}$, r an introduction rule for c and r' an elimination rule for c , we have

- an **introduction term** $\{\bar{t}; \overline{\lambda x : A.t}\}_r$
- an **elimination term** $t \cdot_{r'} [\bar{t}; \overline{\lambda x : A.t}]$



Curry-Howard typing rules

The terms are *typed* using the following derivation rules.

$$\begin{array}{c}
 \frac{}{\Gamma \vdash x_i : A_i} \text{ if } x_i : A_i \in \Gamma \\
 \\
 \frac{\dots \Gamma \vdash p_j : A_j \dots \quad \dots \Gamma, y_i : A_i \vdash q_i : \Phi \dots}{\Gamma \vdash \{\bar{p} ; \overline{\lambda y : A. q}\}_r : \Phi} \text{ in} \\
 \\
 \frac{\Gamma \vdash t : \Phi \quad \dots \Gamma \vdash p_k : A_k \dots \quad \dots \Gamma, y_\ell : A_\ell \vdash q_\ell : D}{\Gamma \vdash t \cdot_r [\bar{p} ; \overline{\lambda y : A. q}] : D} \text{ el}
 \end{array}$$

Here, \bar{p} is the sequence of terms $p_1, \dots, p_{m'}$ for all the 1-entries in the truth table, and $\overline{\lambda y : A. q}$ is the sequence of terms $\lambda y_1 : A_1. q_1, \dots, \lambda y_m : A_m. q_m$ for all the 0-entries in the truth table.



Reductions on terms for detours

Term reduction rules that correspond to **detour conversions**.

- For simplicity we write the “matching cases” as last term of the sequence.
- For the $j = \ell$ case, that is, $p_j : A_j$ and $y_\ell : A_\ell$ with $A_j = A_\ell$:

$$\{\overline{p}, \overline{p_j} ; \overline{\lambda x. q}\} \cdot [\overline{s} ; \overline{\lambda y. r}, \overline{\lambda y_\ell. r_\ell}] \longrightarrow_a r_\ell[y_\ell := p_j]$$

- For the $i = k$ case, that is, $x_i : A_i$ and $s_k : A_k$ with $A_i = A_k$:

$$\{\overline{p} ; \overline{\lambda x. q}, \overline{\lambda x_i. q_i}\} \cdot [\overline{s}, \overline{s_k} ; \overline{\lambda y. r}] \longrightarrow_a q_i[x_i := s_k] \cdot [\overline{s}, \overline{s_k} ; \overline{\lambda y. r}]$$

$\overline{p}, \overline{p_j}$ should be understood as a sequence $p_1, \dots, p_j, \dots, p_{m'}$, where the p_j that matches the r_ℓ in $\overline{\lambda y. r}, \overline{\lambda y_\ell. r_\ell}$ has been singled out.

NB There is always (at least one) **matching case**, because intro/elim rules comes from different lines in the truth table.



Example: reductions for \wedge -terms

The rules for conjunction are as follows.

$$\frac{\vdash t : A \wedge B \quad x : A \vdash p : D \quad y : B \vdash q : D}{\vdash t \cdot \hat{\wedge}_1 [; \lambda x.p, \lambda y.q] : D}$$

$$\frac{\vdash t : A \wedge B \quad \vdash a : A \quad y : B \vdash q : D}{\vdash t \cdot \hat{\wedge}_2 [a ; \lambda y.q] : D}$$

$$\frac{\vdash t : A \wedge B \quad x : A \vdash p : D \quad \vdash b : B}{\vdash t \cdot \hat{\wedge}_3 [b ; \lambda x.p] : D}$$

$$\frac{\vdash a : A \quad \vdash b : B}{\vdash \{ a, b ; \}^\wedge : A \wedge B}$$

The reduction rules are

$$\{ a, b ; \}^\wedge \cdot \hat{\wedge}_1 [; \lambda x.A.p, \lambda y.B.q] \longrightarrow_a p[x := a]$$

$$\{ a, b ; \}^\wedge \cdot \hat{\wedge}_1 [; \lambda x.A.p, \lambda y.B.q] \longrightarrow_a q[y := b]$$

$$\{ a, b ; \}^\wedge \cdot \hat{\wedge}_2 [a' ; \lambda y.B.q] \longrightarrow_a q[y := b]$$

$$\{ a, b ; \}^\wedge \cdot \hat{\wedge}_3 [b' ; \lambda x.A.p] \longrightarrow_a p[x := a]$$

From the first two cases, we see that the Church-Rosser property (confluence) is lost.



Example: reductions for \rightarrow -terms

The rules for implication are as follows.

$$\frac{x : A \vdash p : A \rightarrow B \quad y : B \vdash q : A \rightarrow B}{\vdash \{ ; \lambda x.p, \lambda y.q \}_1^{\rightarrow} : A \rightarrow B}$$

$$\frac{\vdash t : A \rightarrow B \quad \vdash a : A \quad z : B \vdash r : D}{\vdash t \cdot^{\rightarrow} [a ; \lambda z.r] : D}$$

$$\frac{x : A \vdash p : A \rightarrow B \quad \vdash b : B}{\vdash \{ b ; \lambda x.p \}_2^{\rightarrow} : A \rightarrow B}$$

$$\frac{\vdash a : A \quad \vdash b : B}{\vdash \{ a, b ; \}_3^{\rightarrow} : A \rightarrow B}$$

The reduction rules are

$$\begin{aligned} \{ ; \lambda x:A.p, \lambda y:B.q \}_1^{\rightarrow} \cdot^{\rightarrow} [a ; \lambda z:B.r] &\longrightarrow_a p[x := a] \cdot^{\rightarrow} [a ; \lambda z:B.r] \\ \{ b ; \lambda x:A.p \}_2^{\rightarrow} \cdot^{\rightarrow} [a ; \lambda z:B.r] &\longrightarrow_a r[z := b] \\ \{ b ; \lambda x:A.p \}_2^{\rightarrow} \cdot^{\rightarrow} [a ; \lambda z:B.r] &\longrightarrow_a p[x := a] \cdot^{\rightarrow} [a ; \lambda z:B.r] \\ \{ a', b ; \}_3^{\rightarrow} \cdot^{\rightarrow} [a ; \lambda z:B.r] &\longrightarrow_a r[z := b] \end{aligned}$$

In the first and the third case, we see that the elimination remains.



Reductions on terms for permutations

We add the following reduction rules for **permutation conversions**.

$$(t \cdot_r [\bar{p} ; \overline{\lambda x. q}]) \cdot_{r'} [\bar{s} ; \overline{\lambda y. r}] \longrightarrow_b t \cdot_r [\bar{p} ; \overline{\lambda x. (q \cdot_{r'} [\bar{s} ; \overline{\lambda y. r}])}]$$

Here, $\overline{\lambda x. (q \cdot [\bar{s} ; \overline{\lambda y. r}])}$ should be understood as a sequence $\lambda x_1. q_1, \dots, \lambda x_m. q_m$ where each q_j is replaced by $q_j \cdot_{r'} [\bar{s} ; \overline{\lambda y. r}]$.



Example of permutation reduction on terms

$$\frac{\frac{\vdash t : A \vee B \quad x : A \vdash p : C \rightarrow D \quad y : B \vdash q : C \rightarrow D}{\vdash t \cdot^\vee [; \lambda x.p, \lambda y.q] : C \rightarrow D} \quad \vdash c : C \quad z : D \vdash r : E}{\vdash t \cdot^\vee [; \lambda x.p, \lambda y.q] \cdot^\rightarrow [c ; \lambda z.r] : E}$$

The two consecutive elimination rules can be permuted. The term reduces as follows

$$t \cdot^\vee [; \lambda x.p, \lambda y.q] \cdot^\rightarrow [c ; \lambda z.r] \longrightarrow_b t \cdot^\vee [; \lambda x.p \cdot^\rightarrow [c ; \lambda z.r], \lambda y.q \cdot^\rightarrow [c ; \lambda z.r]]$$



Optimized reductions and optimized terms

The usual “pairing” is given by the introduction rule: $\{a, b ; \}^\wedge$.
 For elimination, we want to have the “projection” rules:

$$\frac{\vdash t : A \wedge B}{\vdash \pi_1 t : A} \qquad \frac{\vdash t : A \wedge B}{\vdash \pi_2 t : B}$$

We can define

$$\pi_1 t := t \cdot \hat{\cdot}_1 [; \lambda x^A . x, \lambda z^B . t \cdot \hat{\cdot}_3 [z ; \lambda x^A . x]]$$

which has the following reductions.

$$\begin{aligned} \pi_1 \{a, b ; \}^\wedge &= \{a, b ; \}^\wedge \cdot \hat{\cdot}_1 [; \lambda x^A . x, \lambda z^B . \{a, b ; \}^\wedge \cdot \hat{\cdot}_3 [z ; \lambda x^A . x]] \\ &\longrightarrow_a a \end{aligned}$$

$$\begin{aligned} \pi_1 \{a, b ; \}^\wedge &= \{a, b ; \}^\wedge \cdot \hat{\cdot}_1 [; \lambda x^A . x, \lambda z^B . \{a, b ; \}^\wedge \cdot \hat{\cdot}_3 [z ; \lambda x^A . x]] \\ &\longrightarrow_a \{a, b ; \}^\wedge \cdot \hat{\cdot}_3 [b ; \lambda x^A . x] \\ &\longrightarrow_a a \end{aligned}$$



Another \wedge -elimination rule

The **parallel \wedge -elimination** rule (“general elimination rule”) studied by Schroeder-Heister and Von Plato is:

$$\frac{\Gamma \vdash t : A \wedge B \quad \Gamma, x : A, y : B \vdash q : D}{\Gamma \vdash t \cdot^{\text{par}} [\lambda x, y. q] : D} \wedge\text{-el}$$

The reduction for this rule is as follows.

$$\{a, b ; \} \cdot^{\text{par}} [\lambda x, y. q] \longrightarrow_{\text{par}} q[x := a, y := b].$$

We can define the parallel \wedge -elimination rule

$$t \cdot^{\text{par}} [\lambda x, y. q] := t \cdot^{\wedge}_1 [; \lambda x. p, \lambda z^B. t \cdot^{\wedge}_3 [z ; \lambda x. p]],$$

$$\text{where } p = t \cdot^{\wedge}_1 [; \lambda u^A. t \cdot^{\wedge}_2 [u ; \lambda y. q], \lambda y. q]$$

Then

$$\{a, b ; \} \cdot^{\text{par}} [\lambda x, y. q] \longrightarrow_a^+ q[x := a, y := b].$$



Relating optimized reductions to the original reductions

- Usually, when studying constructive natural deduction, one uses variants of the **optimized rules** for natural deduction.
- For the **optimized rules**, there is also a straightforward definition of **proof-terms** and of the **reduction** relations \longrightarrow_a and \longrightarrow_b .
- Proposition: The proof-terms for the optimized rules can be defined in terms of the terms for the full calculus.
- Proposition: The reduction rules for the optimized proof terms are an instance of reductions in the full calculus (often multi-step).
- So: Strong Normalization for the optimized calculus follows from Strong Normalization for the full calculus.



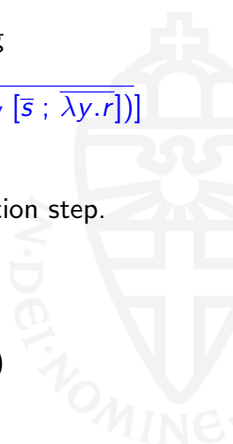
Normalization

THEOREM The reduction \longrightarrow_b is strongly normalizing

$$(t \cdot_r [\bar{p}; \overline{\lambda x. q}]) \cdot_{r'} [\bar{s}; \overline{\lambda y. r}] \longrightarrow_b t \cdot_r [\bar{p}; \overline{\lambda x. (q \cdot_{r'} [\bar{s}; \overline{\lambda y. r}])}]$$

PROOF The measure $| - |$ decreases with every reduction step.

$$\begin{aligned} |x| &:= 1 \\ |\{\bar{p}; \overline{\lambda y. q}\}| &:= \Sigma|p_i| + \Sigma|q_j| \\ |t \cdot [\bar{s}; \overline{\lambda y. u}]| &:= |t|(2 + \Sigma|s_k| + \Sigma|u_\ell|) \end{aligned}$$





Normalization

THEOREM The reduction \rightarrow_a is strongly normalizing.

$$\{\overline{p}, \overline{p_j}; \overline{\lambda x. q}\} \cdot [\overline{s}; \overline{\lambda y. r}, \overline{\lambda y_\ell. r_\ell}] \rightarrow_a r_\ell[y_\ell := p_j]$$

(for the $j = \ell$ case, $p_j : A_j$ and $y_\ell : A_\ell$ with $A_j = A_\ell$)

$$\{\overline{p}; \overline{\lambda x. q}, \overline{\lambda x_i. q_i}\} \cdot [\overline{s}, \overline{s_k}; \overline{\lambda y. r}] \rightarrow_a q_i[x_i := s_k] \cdot [\overline{s}, \overline{s_k}; \overline{\lambda y. r}]$$

(for the $i = k$ case, $x_i : A_i$ and $s_k : A_k$ with $A_i = A_k$)

PROOF We adapt the saturated sets method of Tait.

COROLLARY the combination \rightarrow_{ab} is weakly normalizing.

Basically: take the \rightarrow_b -normal-form and then contract the innermost \rightarrow_a -redex of **highest rank**. (This generalizes the Gandy-Turing SN proof for simple type theory, $\lambda \rightarrow$.)



Strong Normalization

Recently (master thesis of Iris van der Giessen) we have obtained a proof of Strong Normalization for general IPC_C .

Method (generalizing a proof by Philippe De Groote)

- Define a “double negation” translation from IPC_C formulas to $\lambda \rightarrow$ -types.
- Define a reduction preserving “CPS” translation from IPC_C terms to $\lambda \rightarrow$ -parallel.
($\lambda \rightarrow$ extended with $[M_1, \dots, M_n] : A$ if $M_i : A$ for $1 \leq i \leq n$.)
- Prove SN for $\lambda \rightarrow$ -parallel.



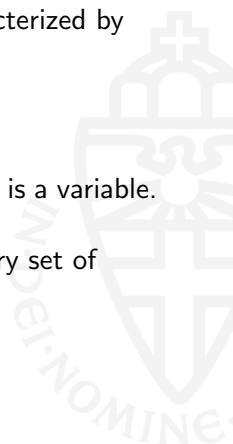
Consequences of Normalization

The set of **terms in normal form** of IPC_C , NF is characterized by the following inductive definition.

- $x \in \text{NF}$ for every variable x ,
- $\{\bar{p} ; \overline{\lambda y. q}\} \in \text{NF}$ if all p_i and q_j are in NF,
- $x \cdot [\bar{p} ; \overline{\lambda y. q}] \in \text{NF}$ if all p_i and q_j are in NF and x is a variable.

As corollaries of Normalization we have, for an arbitrary set of connectives:

- subformula property
- consistency of the logic
- decidability of the logic





The Sheffer stroke or NAND connective [I]

The truth table for $\text{nand}(A, B)$, which we write as $A \uparrow B$ is as follows.

A	B	$A \uparrow B$
0	0	1
0	1	1
1	0	1
1	1	0

From this we derive the following optimized rules.

$\frac{A \vdash A \uparrow B}{\vdash A \uparrow B} \uparrow\text{-inl}$	$\frac{B \vdash A \uparrow B}{\vdash A \uparrow B} \uparrow\text{-inr}$	$\frac{\vdash A \uparrow B \quad \vdash A \quad \vdash B}{\vdash D} \uparrow\text{-el}$
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The Sheffer stroke or NAND connective [II]

The usual connectives can be defined in terms of nand.

$$\begin{aligned} \neg A &:= A \uparrow A \\ A \vee B &:= (A \uparrow A) \uparrow (B \uparrow B) \\ A \wedge B &:= (A \uparrow B) \uparrow (A \uparrow B) \\ A \rightarrow B &:= A \uparrow (B \uparrow B) \end{aligned}$$

This gives rise to an embedding $(-)^{\uparrow}$ of intuitionistic proposition logic \vdash_i into the nand-logic \vdash_{\uparrow} .

PROPOSITION For A a formula in proposition logic,

$$\vdash_i \neg\neg A \quad \iff \quad \vdash_{\uparrow} A^{\uparrow}.$$





The Sheffer stroke or NAND connective [III]

The proof-terms for nand-logic are

$$\frac{x : A \vdash p : A \uparrow B}{\vdash \{ ; \lambda x^A . p \}^\uparrow : A \uparrow B} \quad \frac{y : B \vdash q : A \uparrow B}{\vdash \{ ; \lambda y^B . q \}^\uparrow : A \uparrow B}$$

$$\frac{\vdash t : A \uparrow B \quad \vdash a : A \quad \vdash b : B}{\vdash t \cdot^\uparrow [a, b ;] : D}$$

with reduction rules

$$\{ ; \lambda x^A . p \}^\uparrow \cdot^\uparrow [a, b ;] \longrightarrow_a p[x := a] \cdot^\uparrow [a, b ;]$$

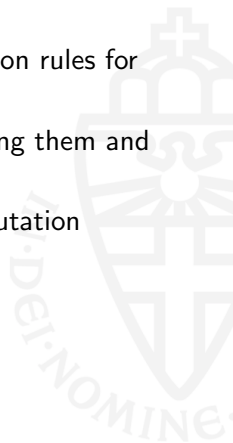
$$\{ ; \lambda y^B . q \}^\uparrow \cdot^\uparrow [a, b ;] \longrightarrow_a q[y := b] \cdot^\uparrow [a, b ;]$$





Conclusions

- Simple general way to derive constructive deduction rules for (new) connectives.
- Study connectives “in isolation”. (Without defining them and without using other connectives.)
- General definition of detour conversion and permutation conversion.
- General proof-as-terms interpretation.
- General (Strong) Normalization proof.



Future work and Related

- Meaning of the new connectives as inductive data types.
- Study conditions for the set of rules to be Church-Rosser.
- General definition of classical detour/permutation conversion
- Relation with other term calculi for classical logic: subtraction logic, $\lambda\mu$ (Parigot), $\bar{\lambda}\mu\tilde{\mu}$ (Curien, Herbelin).

Related work:

- Roy Dyckhoff, Peter Milne, Jan von Plato and Sara Negri, Peter Schroeder-Heister, ...
- “Harmony” in logic (following Prawitz)