Normalisation for general constructive propositional logic

Herman Geuvers
(joint work with Tonny Hurkens and Iris van der Giessen)

Institute for Computing and Information Science
Radboud University
Nijmegen, NL

Types 2018
University of Minho
June 20, 2018
Constructively the meaning of a connective is fixed by explaining what a proof is that involves the connective. Basically, this explains the introduction rule(s) for each connective, from which the elimination rules follow (Prawitz). By analysing constructive proofs we then also get

- consistency (from proof normalization),
- decidability (from the subformula property),
- Curry-Howard proofs-as-terms (and propositions-as-types) allowing to study normalization as term-reduction.
Our work / Overview of the talk

- Derive **constructive** natural deduction rules for a connective from its **truth table** definition.
  - Gives natural deduction rules for a connective “in isolation”
  - Also gives constructive rules for connectives that haven’t been studied constructively so far, like **nand** and **if-then-else**.

- Curry-Howard: give proof-terms for natural deductions

- Define proof-normalization as term-reduction
  ⇒ A general notion of **detour conversion** (cut-elimination) and **permutation conversion** for these constructive connectives.

- Weak normalization

- Strong normalization
Standard form for natural deduction rules

\[
\Gamma \vdash A_1 \quad \ldots \quad \Gamma \vdash A_n \quad \Gamma, B_1 \vdash D \quad \ldots \quad \Gamma, B_m \vdash D \\
\hline
\Gamma \vdash D
\]

If the conclusion of a rule is \( \Gamma \vdash D \), then the hypotheses of the rule can be of one of two forms:

1. \( \Gamma, B \vdash D \). We are given extra data \( B \) to prove \( D \) from \( \Gamma \). We call \( B \) a **Case**.

2. \( \Gamma \vdash A \). instead of proving \( D \) from \( \Gamma \), we now need to prove \( A \) from \( \Gamma \). We call \( A \) a **Lemma**.

One obvious advantage: we don’t have to give \( \Gamma \) explicitly, as it can be retrieved:

\[
\vdash A_1 \quad \ldots \quad \vdash A_n \quad B_1 \vdash D \quad \ldots \quad B_m \vdash D \\
\hline
\vdash D
\]
Let $c$ be an $n$-ary connective $c$ with truth table $t_c$. Each row of $t_c$ gives rise to an elimination rule or an introduction rule for $c$. (We write $\varphi = c(A_1, \ldots, A_n).$)

**Elimination**

\[
\frac{A_1 \ldots A_n}{p_1 \ldots p_n} \varphi \quad \implies \quad \vdash \varphi \ldots \vdash A_j \quad (\text{if } p_j = 1) \ldots A_i \vdash D \quad (\text{if } p_i = 0) \ldots \vdash D
\]

**Introduction**

\[
\frac{A_1 \ldots A_n}{q_1 \ldots q_n} \varphi \quad \implies \quad \ldots \vdash A_j \quad (\text{if } q_j = 1) \ldots A_i \vdash \varphi \quad (\text{if } q_i = 0) \ldots \vdash \varphi
\]

This is the **constructive** introduction rule; there is also a **classical** introduction rule, which we don’t discuss now.
Definition of the logics

Given a set of connectives \( C := \{c_1, \ldots, c_n\} \), the constructive natural deduction systems for \( C \), IPC\(_C\), has the following rules.

- The **axiom rule**

  \[
  \frac{}{\Gamma \vdash A} \quad \text{(if } A \in \Gamma) \]

- The **constructive introduction rules** for the connectives in \( C \), as derived from the truth table.

- The **elimination rules** for the connectives in \( C \), as derived from the truth table.
Examples

Derivation rules for $\land$ (3 elim rules and one intro rule):

<table>
<thead>
<tr>
<th>A</th>
<th>B</th>
<th>A $\land$ B</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>

$\vdash A \land B$ $A \vdash D$ $B \vdash D$

$\vdash D \quad \land$-el$_a$

$\vdash A \land B$ $\vdash A$ $B \vdash D$

$\vdash D \quad \land$-el$_c$

$\vdash A \land B$ $\vdash A \quad B \vdash D$

$\vdash D \quad \land$-in

$\vdash A \quad B$

$\vdash A \land B$ $\land$-el$_b$

• These rules can be shown to be equivalent to the well-known derivation rules.
• These rules can be optimized to 3 rules.
Examples

Rules for \(\neg\): 1 elimination rule and 1 introduction rule.

\[
\begin{array}{c|c}
A & \neg A \\
\hline
0 & 1 \\
1 & 0 \\
\end{array}
\]

Derivation rules:

\[
\frac{\neg A}{\vdash D} \quad \frac{A}{\vdash A} \quad \frac{A}{\vdash \neg A} \\
\text{\(-el\)} \quad \text{\(-el\)} \quad \text{\(-in\)}
\]
The well-known constructive connectives

The optimised rules for $\land$, $\neg$, $\lor$, $\rightarrow$, $\top$ and $\bot$ we obtain are:

\[
\begin{align*}
\frac{\vdash A \quad \vdash B}{\vdash A \land B} & \quad \land\text{-in} \\
\frac{\vdash A \land B}{\vdash A} & \quad \land\text{-el}_1 \\
\frac{\vdash A \land B}{\vdash B} & \quad \land\text{-el}_2 \\
\frac{\vdash A \lor B \quad A \vdash D \quad B \vdash D}{\vdash D} & \quad \lor\text{-el} \\
\frac{\vdash A}{\vdash A \lor B} & \quad \lor\text{-in}_1 \\
\frac{\vdash B}{\vdash A \lor B} & \quad \lor\text{-in}_2 \\
\frac{\vdash A \rightarrow B \quad \vdash A}{\vdash B} & \quad \rightarrow\text{-el} \\
\frac{\vdash B}{\vdash A \rightarrow B} & \quad \rightarrow\text{-in}_1 \\
\frac{A \vdash A \rightarrow B}{\vdash A \rightarrow B} & \quad \rightarrow\text{-in}_2 \\
\frac{\vdash \neg A \quad \vdash A}{\vdash D} & \quad \neg\text{-el} \\
\frac{A \vdash \neg A}{\vdash \neg A} & \quad \neg\text{-in} \\
\frac{\vdash \top}{\vdash \top} & \quad \top\text{-in} \\
\frac{\vdash \bot}{\vdash \bot} & \quad \bot\text{-el} \\
\end{align*}
\]
Sheffer stroke or NAND connective [I]

The truth table for \( \text{nand}(A, B) \), which we write as \( A \uparrow B \), is as follows.

<table>
<thead>
<tr>
<th></th>
<th></th>
<th>( A \uparrow B )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>0</td>
</tr>
</tbody>
</table>

From this we derive the following optimized rules.

\[
\begin{align*}
A \vdash A \uparrow B & \quad \text{\(-inl\)} \\
\vdash A \uparrow B & \quad \text{\(-inl\)} \\
B \vdash A \uparrow B & \quad \text{\(-inr\)} \\
\vdash A \uparrow B & \quad \text{\(-inr\)} \\
\vdash A \uparrow B & \quad \text{\(-el\)} \\
\vdash A & \quad \vdash \quad \vdash B & \quad \vdash D
\end{align*}
\]
Sheffer stroke or NAND connective [II]

The usual connectives can be defined in terms of nand.

\[ \neg A := A \uparrow A \]
\[ A \lor B := (A \uparrow A) \uparrow (B \uparrow B) \]
\[ A \land B := (A \uparrow B) \uparrow (A \uparrow B) \]
\[ A \rightarrow B := A \uparrow (B \uparrow B) \]

This gives rise to an embedding \((\neg)\uparrow\) of intuitionistic proposition logic \(\vdash\) into the nand-logic \(\vdash\uparrow\).

**Proposition** For \(A\) a formula in proposition logic,

\[ \vdash \neg \neg A \iff \vdash \uparrow (A\uparrow). \]
Detours (cuts) in constructive logic

Remember that the rules for $c$ arise from rows in the truth table $t_c$:

<table>
<thead>
<tr>
<th>$A_1$</th>
<th>$\ldots$</th>
<th>$A_n$</th>
<th>$c(A_1, \ldots, A_n)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$p_1$</td>
<td>$\ldots$</td>
<td>$p_n$</td>
<td>0</td>
</tr>
<tr>
<td>$q_1$</td>
<td>$\ldots$</td>
<td>$q_n$</td>
<td>1</td>
</tr>
</tbody>
</table>

**Definition** A detour convertibility is a pattern of the following form, where $\varphi = c(A_1, \ldots, A_n)$.

\[\begin{array}{c}
\vdots \\
\Sigma_j \\
\Gamma \vdash A_j \\
\vdots \\
\Sigma_i \\
\Gamma, A_i \vdash \varphi \\
\vdots \\
\Gamma \vdash \varphi \\
\end{array}\]

\[\begin{array}{c}
\vdots \\
\Pi_k \\
\Gamma \vdash A_k \\
\vdots \\
\Pi_\ell \\
\Gamma, A_\ell \vdash D \\
\vdots \\
\Gamma \vdash D \\
\end{array}\]

- $q_j = 1$ for $A_j$ and $q_i = 0$ for $A_i$
- $p_k = 1$ for $A_k$ and $p_\ell = 0$ for $A_\ell$
The *elimination of a detour* is defined by replacing the deduction pattern by another one. If $A_j = A_\ell$ (for some $j, \ell$), replace

\[
\begin{array}{c}
\Sigma_j \\
\Gamma \vdash A_j \\
\end{array}
\quad \cdots
\quad \begin{array}{c}
\Sigma_i \\
\Gamma, A_i \vdash \varphi \\
\end{array}
\quad \cdots
\]

in

\[
\begin{array}{c}
\Gamma \vdash \varphi \\
\end{array}
\]

by

\[
\begin{array}{c}
\Sigma_j \\
\Gamma \vdash A_j \\
\end{array}
\quad \cdots
\quad \begin{array}{c}
\Sigma_j \\
\Gamma \vdash A_j \\
\end{array}
\quad \begin{array}{c}
\Pi_\ell \\
\Gamma \vdash D \\
\end{array}
\quad \cdots
\]

\[
\begin{array}{c}
\Pi_k \\
\Gamma \vdash A_k \\
\end{array}
\quad \cdots
\quad \begin{array}{c}
\Gamma, A_\ell \vdash D \\
\end{array}
\quad \text{el}
\]

\[
\begin{array}{c}
\Gamma \vdash D \\
\end{array}
\]
Eliminating a detour (detour conversion) (II)

If \( A_i = A_k \) (for some \( i, k \)), replace

\[
\begin{align*}
\Sigma_j & \quad \Gamma \vdash A_j \\
\Sigma_i & \quad \Gamma, A_i \vdash \varphi \\
\hline
\Gamma & \vdash \varphi \\
\end{align*}
\]

by

\[
\begin{align*}
\Pi_k & \quad \Gamma \vdash A_k \\
\dots & \quad \Gamma \vdash A_k \\
\Sigma_i & \quad \Gamma \vdash \varphi \\
\hline
\Gamma & \vdash \varphi \\
\end{align*}
\]

\[
\begin{align*}
\Pi_k & \quad \Gamma \vdash A_k \\
\Pi_\ell & \quad \Gamma, A_\ell \vdash D \\
\hline
\Gamma & \vdash D \\
\end{align*}
\]
Observation

\[ \Sigma_j \quad \cdots \quad \Gamma \vdash A_j \cdots \quad \Sigma_i \quad \Gamma, A_i \vdash \varphi \quad \cdots \quad \in \quad \Pi_k \quad \cdots \quad \Gamma \vdash \varphi \quad \cdots \quad \Pi_{\ell} \quad \Gamma, A_{\ell} \vdash D \quad \cdots \quad \text{el} \]

- There can be several “matching” \((i, k)\) or \((j, \ell)\) pairs.
- So: detour conversion ("\(\beta\)-rule") is non-deterministic in general.
Permutation convertibility: Definition

Let \( c \) and \( c' \) be connectives of arity \( n \) and \( n' \), with elimination rules \( r \) and \( r' \) respectively. A permutation convertibility in a derivation is a pattern of the following form, where \( \Phi = c(B_1, \ldots, B_n) \), \( \Psi = c'(A_1, \ldots, A_{n'}) \).

\[
\vdash \Sigma_j \quad \vdash \Sigma_i \\
\vdash \Psi \quad \vdash A_j \quad \ldots \quad \vdash A_i \quad \vdash \Phi \quad \ldots \\
\vdash \Phi \quad \vdash A_1 \quad \ldots \quad \vdash D \quad \vdash B_k \quad \ldots \\
\vdash B_\ell \quad \vdash D \\
\vdash D \\
el_r \quad el_{r'} \\
\]

- \( A_j \) ranges over all propositions that have a 1 in the truth table of \( c' \); \( A_i \) ranges over all propositions that have a 0,
- \( B_k \) ranges over all propositions that have a 1 in the truth table of \( c \); \( B_\ell \) ranges over all propositions that have a 0.
Permutation conversion

The permutation conversion is defined by replacing the derivation pattern on the previous slide by

\[
\vdash \psi \ldots \vdash A_j \quad \ldots \quad A_i \vdash \phi \quad \ldots \quad A_i \vdash B_k \quad \ldots \quad A_i, B_{\ell} \vdash D \quad \ldots \quad A_i \vdash D
\]

\[
\vdash D
\]

This gives rise to copying of sub-derivations: for every \( A_i \) we copy the sub-derivations \( \Pi_1, \ldots, \Pi_n \).
We define rules for the judgment $\Gamma \vdash t : A$, where

- $A$ is a formula,
- $\Gamma$ is a set of declarations $\{x_1 : A_1, \ldots, x_m : A_m\}$, where the $A_i$ are formulas and the $x_i$ are term-variables,
- $t$ is a proof-term:

$$ t ::= x \mid \{ t ; \lambda x : A. t \} r \mid t \cdot r [ t ; \lambda x : A. t ] $$

where $x$ ranges over variables and $r$ ranges over the rules.

For a connective $c \in C$, $r$ an introduction rule for $c$ and $r'$ an elimination rule for $c$, we have

- an introduction term $\{ t ; \lambda x : A. t \} r$
- an elimination term $t \cdot r' [ t ; \lambda x : A. t ]$
Let $\Phi = c(A_1, \ldots, A_n)$ and $r$ a rule for $c$.

\[
\begin{array}{c}
\text{if } \Gamma \vdash x_i : A_i \in \Gamma \\
\Gamma \vdash x_i : A_i \\
\quad \ldots \Gamma \vdash p_j : A_j \ldots \ldots \Gamma, y_i : A_i \vdash q_i : \Phi \ldots \\
\Gamma \vdash \{\overline{p}; \lambda y : A.q\}_r : \Phi \\
\Gamma \vdash t : \Phi \ldots \Gamma \vdash p_k : A_k \ldots \ldots \Gamma, y_\ell : A_\ell \vdash q_\ell : D \\
\Gamma \vdash t \cdot_r [\overline{p}; \lambda y : A.q] : D
\end{array}
\]

Here, $\overline{p}$ is the sequence of terms $p_1, \ldots, p_m$ for all the 1-entries in rule $r$ of the truth table, and $\overline{\lambda y : A.q}$ is the sequence of terms $\lambda y_1 : A_1.q_1, \ldots, \lambda y_m : A_m.q_m$ for all the 0-entries in $r$. 
Reductions on terms for detours

Term reduction rules that correspond to detour conversions.

- For simplicity we write the “matching cases” as last term of the sequence.
- For the $A_j = A_\ell$ case, that is, $p_j : A_j$ and $y_\ell : A_\ell$ with $A_j = A_\ell$:
  
  $$\{ p, p_j ; \lambda x . q \} \cdot [s ; \lambda y . r, \lambda y_\ell . r_\ell] \rightarrow_a r_\ell[y_\ell := p_j]$$

- For the $A_i = A_k$ case, that is, $x_i : A_i$ and $s_k : A_k$ with $A_i = A_k$:
  
  $$\{ p ; \lambda x . q, \lambda x_i . q_i \} \cdot [s, s_k ; \lambda y . r] \rightarrow_a q_i[x_i := s_k] \cdot [s, s_k ; \lambda y . r]$$

$p, p_j$ should be understood as a sequence $p_1, \ldots, p_j, \ldots, p_m$, where the $p_j$ that matches the $r_\ell$ in $\lambda y . r, \lambda y_\ell . r_\ell$ has been singled out.

**NB** There is always (at least one) matching case, because intro/elim rules comes from different lines in the truth table.
We add the following reduction rules for permutation conversions.

\[(t \cdot_r [p ; \lambda x. q]) \cdot_{r'} [s ; \lambda y. r] \rightarrow_b t \cdot_r [p ; \lambda x. (q \cdot_{r'} [s ; \lambda y. r])]\]

Here, \(\lambda x.(q \cdot [s ; \lambda y. r])\) should be understood as a sequence \(\lambda x_1.q_1, \ldots, \lambda x_m.q_m\) where each \(q_j\) is replaced by \(q_j \cdot_{r'} [s ; \lambda y. r]\).
On optimized terms, one can also, in a canonical way, define detour conversion $\rightarrow_a$ and permutation conversion $\rightarrow_b$.

Detour reduction on optimised terms translates to (multi-step) detour reduction on the full terms.

So, strong normalization on optimised terms follows from strong normalization on full terms.

Other well-known rules, like the general elimination rules studied by Schroeder-Heister and Von Plato, can similarly be translated to our full rules.
**Theorem** The reduction $\rightarrow_b$ is strongly normalizing

\[
(t \cdot_r [\overline{p}; \overline{\lambda x.q}]) \cdot_{r'} [\overline{s}; \overline{\lambda y.r}] \rightarrow_b t \cdot_r [\overline{p}; \overline{\lambda x.(q \cdot_{r'} [\overline{s}; \overline{\lambda y.r}])}]
\]

**Proof** The measure $|\_|$ decreases with every reduction step.

\[
|x| := 1 \\
|\{\overline{p}; \overline{\lambda y.q}\}| := \Sigma |p_i| + \Sigma |q_j| \\
|t \cdot [\overline{s}; \overline{\lambda y.u}]| := |t|(2 + \Sigma |s_k| + \Sigma |u_\ell|)
\]
**Theorem** The reduction $\rightarrow_a$ is strongly normalizing.

$$\{p, p_j ; \lambda x. q\} \cdot [s ; \lambda y.r, \lambda y_\ell.r_\ell] \rightarrow_a r_\ell[y_\ell := p_j]$$

(for the $A_j = A_\ell$ case, $p_j : A_j$ and $y_\ell : A_\ell$ with $A_j = A_\ell$)

$$\{p ; \lambda x.q, \lambda x_i.q_i\} \cdot [s, s_k ; \lambda y.r] \rightarrow_a q_i[x_i := s_k] \cdot [s, s_k ; \lambda y.r]$$

(for the $A_i = A_k$ case, $x_i : A_i$ and $s_k : A_k$ with $A_i = A_k$)

**Proof** We adapt the saturated sets method of Tait.

**Corollary** the combination $\rightarrow_{ab}$ is weakly normalizing. Basically: take the $\rightarrow_b$-normal-form and then contract the innermost $\rightarrow_a$-redex of highest rank. (This generalizes the Gandy-Turing SN proof for simple type theory, $\lambda \rightarrow$.)
New: we have obtained a proof of Strong Normalization for general $\text{IPC}_C$.

Rough outline of the proof (generalizing a proof of SN for IPC by Philippe De Groote):

- Define a “double negation” translation from $\text{IPC}_C$ formulas to $\lambda \rightarrow$-types.
- Define a reduction preserving “CPS” translation from $\text{IPC}_C$ terms to $\lambda \rightarrow$-parallel.
  $(\lambda \rightarrow$ extended with $[M_1, \ldots, M_n] : A$ if $M_i : A$ for $1 \leq i \leq n.)$
- Prove SN for $\lambda \rightarrow$-parallel.
\lambda \rightarrow \text{-parallel}

- **Types:** $\sigma ::= o \mid (\sigma \rightarrow \sigma)$
- **Terms:** $M ::= x \mid (M M) \mid (\lambda x.M) \mid [M_1, \ldots, M_n]$ ($n > 1$).
- **Typing rules**

  \[
  \begin{align*}
  \Gamma \vdash M : A \rightarrow B & \quad \Gamma \vdash N : A \\
  \Gamma \vdash MN : B & \quad \Gamma, x : A \vdash M : B \\
  \Gamma \vdash \lambda x.M : A \rightarrow B & \quad \Gamma \vdash M_1 : A \quad \ldots \quad \Gamma \vdash M_n : A \\
  \Gamma \vdash [M_1, \ldots, M_n] : A & \quad \Gamma \vdash x : A
  \end{align*}
  \]

- **Reduction rules:** $(\lambda x.M) N \rightarrow_\beta M[x := N]$ plus

  \[
  [M_1, \ldots, M_n] N \rightarrow_\beta \left[[M_1 N, \ldots, M_n N]\right]
  \]

  SN can be proved by adapting the well-known Tait proof.
Translating formulas to types (outline)

Abbreviate \( \neg A := A \rightarrow o. \)

- For a proposition letter, \( \hat{A} := \neg\neg A. \)
- For \( \Phi = c(A_1, \ldots, A_n) \) with elimination rules \( r_1, \ldots, r_t \)
  \[ \hat{\Phi} := \neg (E_1 \rightarrow \cdots \rightarrow E_t \rightarrow o), \]

where

\[ E_s := \hat{A}_{k_1} \rightarrow \cdots \rightarrow \hat{A}_{k_m} \rightarrow \neg\hat{A}_{l_1} \rightarrow \cdots \rightarrow \neg\hat{A}_{l_{n-m}} \rightarrow o \]

with the \( A_k \) the 1-entries and the \( A_l \) are the 0-entries in the truth table.

For example

\[ \hat{A} \land \hat{B} = \neg (\neg\neg\hat{A} \rightarrow \neg\neg\hat{B} \rightarrow o) \]

\[ \hat{A} \lor \hat{B} = \neg ((\neg\hat{A} \rightarrow \neg\hat{B} \rightarrow o) \rightarrow o) \]
Let \( \Phi = c(A_1, \ldots, A_n) \) have elimination rules \( r_1, \ldots, r_t \).

- \( \widehat{x} := \lambda h. x \ h. \)

- Elimination term:
  \[
  M \cdot_{r_s} \overline{[N ; \lambda x. Q]} := \lambda h. \widehat{M} (\lambda g_1 \ldots g_t. \ g_s. \overline{N}(\lambda x. \widehat{Q} \ h)).
  \]

- Introduction term
  \[
  \{\overline{N ; \lambda y. M}\}_{r} := \lambda h. h\ e_1^h \ldots e_t^h,
  \]

where \( e_s^h \) is the possibly parallel term containing

- \( \overline{\lambda f. f_\ell. \widehat{N}_j} \) for \( \ell \) in rule \( r_s \) and \( j \) with \( A_j = A_\ell \).
- \( \overline{\lambda f. (\lambda y_i. \widehat{M}_i \ h)} \) for \( k \) in rule \( r_s \) and \( i \) with \( A_i = A_k \).
Translating proof-terms to $\lambda \to$-parallel terms (outline)

Using the translation $\hat{M}$ we define a second translation $\hat{\hat{M}}$. (This is a generalization of the CPS translation $\overline{M}$ of Plotkin, that De Groote also uses.)

We can prove

1. If $M \longrightarrow_b N$, then $\hat{\hat{M}} = \hat{\hat{N}}$
2. If $\hat{\hat{M}} \subset K$ ($\hat{\hat{M}}$ is a subterm of $K$), then $M \mapsto \hat{\hat{M}} \subset \exists K'$

From this we can derive Strong Normalization.
Conclusions

- Simple general way to derive constructive deduction rules for (new) connectives.
- Study connectives “in isolation”. (Without other connectives.)
- General definition of detour conversion and permutation conversion.
- General Curry-Howard proofs-as-terms interpretation.
- General Strong Normalization proof.
Future work and Related

- Meaning of the new connectives as inductive data types.
- Study conditions for the set of rules to be Church-Rosser.
- General definition of classical detour/permutation conversion
- Relation with other well-known term calculi for classical logic: subtraction logic (Crolard), $\lambda\mu$ (Parigot), $\bar{\lambda}\mu\bar{\nu}$ (Curien, Herbelin).

Related work:
- Roy Dyckhoff, Peter Milne, Jan von Plato and Sara Negri, Peter Schroeder-Heister, . . .
- “Harmony” in logic (following Prawitz)
Questions?