A type system for Continuation Calculus

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Continuation calculus; motivation

- Simple model for functional computation. (Inspired by $\lambda$-calculus and term rewriting)
- But: no variable binding, no pattern matching
- Treat continuations as the fundamental object (rather than expressions or data)
- Deterministic computation
- Turing complete
- Call-by-name and Call-by-value via function definitions
Continuation calculus; rules

- Infinite set of names, \( N \), usually indicated by a capital.
- Terms: either a name, or two terms combined by a dot.

\[
T ::= N \mid (T.T)
\]

- We write \( a.b.c \) as shorthand for \( (a.b).c \).
- A program \( P \) consists of a set of rules, of the form

\[
N.x_1\ldots x_n \rightarrow t
\]

- \( N \) is a name and \( x_1, \ldots, x_n \) are distinct variables,
- \( t \) is a term over the variables \( x_1, \ldots, x_n \)
- Proviso: for each name \( N \) there is at most one rule. We say that the rule defines the name \( N \).
- Evaluation (reduction) of terms is defined by

if \( N.x_1\ldots x_n \rightarrow t \in P \), then

\[
N.t_1\ldots t_n \rightarrow_P t[t_1/x_1, \ldots, t_n/x_n]
\]
Continuation calculus, remarks

Assume we have a program rule

\[ N.x_1 \ldots x_n \rightarrow t. \]

- A term \( N.t_1 \ldots t_k \) with \( k \neq n \) does not reduce at all.
- There is only head reduction: \( M.(N.t_1 \ldots t_n) \ldots \) does not reduce.
- Reduction is trivially confluent, because at most one step is possible.
- Reduction is not necessarily terminating, e.g.

\[ \text{Omega}.x \rightarrow x.x \]

- There is no pattern matching (as one has e.g. in TRS)
- There is no variable binding (as one has in \( \lambda \)-calculus)
- CC is Turing complete
Continuation calculus, examples

- booleans

  \[\text{True}.x.y \rightarrow x\]
  \[\text{False}.x.y \rightarrow y\]

- natural numbers

  \[\text{Zero}.z.s \rightarrow z\]
  \[\text{Succ}.x.z.s \rightarrow s.x\]

  Interpretation of data as CC-terms:
  \[
  \langle m \rangle := \text{Succ}^m.\text{Zero}
  \]
  which is \(\text{Succ}.(\text{Succ} \ldots \text{Succ}.\text{Zero})\ldots\) \(m\)-times.

- lists

  \[\text{Nil}.n.l \rightarrow n\]
  \[\text{Cons}.x.y.n.l \rightarrow l.x.y\]
Continuation calculus, Scott data types

Compute functions by computing a value and passing it on to the next function (continuation)

Natural numbers

\[
\begin{align*}
\text{Zero} & . z . s \rightarrow z \\
\text{Succ} . x . z . s & \rightarrow s . x
\end{align*}
\]

Idea: A number takes two continuations \( z \) and \( s \) and either
- continues with \( z \) (if the number is \( \text{Zero} \))
- continues with \( s . t \) (if the number is \( \text{Succ} . t \))

The definition of data in CC follows the Scott data types approach in untyped \( \lambda \)-calculus (as opposed to the Church approach):

\[
\begin{align*}
\text{Zero} & \ := \ \lambda z s . z \\
\text{Succ} & \ := \ \lambda x z s . s x
\end{align*}
\]

The Scott approach has case distinction as basic and not iteration.
Continuation calculus, example

Natural numbers

\[
\begin{align*}
\text{Zero}.z.s & \rightarrow z \\
\text{Succ}.x.z.s & \rightarrow s.x
\end{align*}
\]

For addition we want

\[
\text{Add}.\langle m \rangle.\langle p \rangle.r \rightarrow r.\langle m + p \rangle
\]

The algorithm for addition that we implement is basically the following term rewriting system.

\[
\begin{align*}
\text{plus}(0, m) & \rightarrow m \\
\text{plus}(S(p), m) & \rightarrow \text{plus}(p, S(m))
\end{align*}
\]
Continuation calculus, computing addition

Natural numbers

\[
\begin{align*}
\text{Zero}.z.s & \rightarrow z \\
\text{Succ}.x.z.s & \rightarrow s.x
\end{align*}
\]

We want \(\text{Add}.\langle m \rangle.\langle p \rangle.r \rightarrow r.\langle m + p \rangle\)

So we take as program rule for \(\text{Add}\):

\[
\text{Add}.x.y.r \rightarrow x.(r.y).t \text{ with } t \text{ yet to be found}
\]

Because then

\[
\begin{align*}
\text{Add}.\text{Zero}.y.r & \rightarrow \text{Zero}.(r.y).t \rightarrow r.y \\
\text{Add}.(\text{Succ}.x).y.r & \rightarrow \text{Succ}.x.(r.y).t \rightarrow t.x \ ??? \rightarrow \text{Add}.x.(\text{Succ}.y).r
\end{align*}
\]

So take \(t = B.y.r\) with

\[
B.y.r.x \rightarrow \text{Add}.x.(\text{Succ}.y).r
\]
Continuation calculus, call-by-value addition

Natural numbers

\[
\begin{align*}
\text{Zero}.z.s & \rightarrow z \\
\text{Succ}.x.z.s & \rightarrow s.x
\end{align*}
\]

\[
\begin{align*}
\text{Add}.x.y.r & \rightarrow x.(r.y).(B.y.r) \\
B.y.r.x & \rightarrow \text{Add}.x.(\text{Succ}.y).r
\end{align*}
\]

Lemma. For all \( m, p \in \mathbb{N} \): \( \text{Add}.\langle m \rangle.\langle p \rangle.r \rightarrow r.\langle m + p \rangle \)

The function is call-by-value in its first argument:

\( \text{Add}.x.y.r \) first computes \( x \), say \( x = \langle p \rangle \), then it returns \( r.(\text{Succ}^p.y) \)
Continuation calculus, call-by-name addition

Natural numbers

\[ \text{Zero.z.s} \to z \]
\[ \text{Succ.x.z.s} \to s.x \]

Call-by-name addition function:

\[ \text{AddCBN.x.y.z.s} \to x.(y.z.s).(C.y.s) \]
\[ C.y.s.x' \to s.(\text{AddCBN.x'.y}) \]

Then we have

\[ \text{AddCBN.}(\text{Succ.}\langle m \rangle).\langle p \rangle.z.s \to \]
\[ \text{Succ.}\langle m \rangle.\langle \langle p \rangle.z.s \rangle.(C.\langle p \rangle.s) \]
\[ \to C.\langle p \rangle.s.\langle m \rangle \]
\[ \to s.(\text{AddCBN.}\langle m \rangle.\langle p \rangle) \]

which is a normal form.

We need a notion of observational equivalence \( \approx \) to prove

\[ \text{AddCBN.}\langle m \rangle.\langle p \rangle \approx \langle m + p \rangle. \]
Definition: Terms $M$ and $N$ are observationally equivalent under program $P$, when for all extension programs $P' \supseteq P$ and terms $X$:

$$X \cdot M \downarrow_{P'} \iff X \cdot N \downarrow_{P'}$$

where $T \downarrow_{P'}$ denotes that $T$ terminates with the program rules $P'$. Notation: $M \approx_P N$, 
A step back: Typed $\lambda$-calculus for Scott data?

Scott numerals:

$$
\begin{align*}
0 & := \lambda x\ f.x \\
p + 1 & := \lambda x\ f.f\ p \\
S & := \lambda n.\lambda x\ f.f\ n
\end{align*}
$$

To type this we need $\text{nat} = A \rightarrow (\text{nat} \rightarrow A) \rightarrow A$.
In $\lambda 2$, we cannot do this . . . unless we extend it with (positive) recursive types.

$$
\text{nat} := \mu Y.\forall X. X \rightarrow (Y \rightarrow X) \rightarrow X
$$

with rule

$$
\text{nat} = \forall X. X \rightarrow (\text{nat} \rightarrow X) \rightarrow X.
$$

[Abadi, Cardelli, Plotkin 1993]

NB. One does not get iteration (recursion) “for free”.

Types for Scott data in CC

In CC everything happens on the ‘top level’. Call this level \( \bot \).

Types:

\[
T := \bot | X | T \rightarrow T | \mu X. T
\]

under the condition that \( X \) is positive in \( T \).

Type equality rule

\[
\mu X. T = T[\mu X. T/X]
\]

Examples:

\[
\begin{align*}
\text{bool} & := \bot \rightarrow \bot \rightarrow \bot \\
\text{nat} & := \bot \rightarrow (\text{nat} \rightarrow \bot) \rightarrow \bot \\
\text{list}_A & := \bot \rightarrow (A \rightarrow \text{list}_A \rightarrow \bot) \rightarrow \bot 
\end{align*}
\]

\[
\begin{align*}
\text{Zero}.z.s & \rightarrow z \\
\text{Succ}.x.z.s & \rightarrow s.x \\
\text{Nil}.n.l & \rightarrow n \\
\text{Cons}.x.y.n.l & \rightarrow l.x.y
\end{align*}
\]
Typing addition in CC

\[
\text{nat} := \bot \to (\text{nat} \to \bot) \to \bot
\]

\[
\begin{align*}
\text{Zero}.z.s & \to z \\
\text{Succ}.x.z.s & \to s.x \\
\text{AddCBV}.x.y.r & \to x.(r.y).(B.y.r) \\
B.y.r.x & \to \text{AddCBV}.x.(\text{Succ}.y).r
\end{align*}
\]

Then

\[
\text{AddCBV} : \text{nat} \to \text{nat} \to \neg\neg\text{nat}
\]

where \( \neg A \) is defined as \( A \to \bot \).

We can also type the call-by-name addition

\[
\text{AddCBN} : \text{nat} \to \text{nat} \to \text{nat}
\]
Type system for CC judgments

- **program signature** $\Sigma$: finite list of distinct names assigned to types
  $\Sigma = n_1 : A_1, \ldots, n_p : A_p$
- **typing context** $\Gamma$: finite list of distinct variables assigned to types
  $\Gamma = x_1 : A_1, \ldots, x_n : A_n$
- $\Sigma$ gives the types of the names (specific for a program $P$). The context is just a “temporary” set of variables, to define program rules.
- **Two kinds of judgment:**
  1. $\Sigma \vdash P$ to express that, given a program signature $\Sigma$, $P$ is a well-typed program. (So $P$ will consist of program rules.)
  2. $\Gamma \vdash_\Sigma M : A$ to express that the term $M$ with free variables in $\Gamma$ has type $A$, given signature $\Sigma$ and context $\Gamma$. 
Derivation rules

- The derivation rules for typing judgments:
  \[ \begin{align*}
  x : A \in \Gamma & \quad \Rightarrow \quad \Gamma \vdash_{\Sigma} x : A \\
  n : A \in \Sigma & \quad \Rightarrow \quad \Gamma \vdash_{\Sigma} n : A \\
  \Gamma \vdash_{\Sigma} M : A \rightarrow B & \quad \Gamma \vdash_{\Sigma} N : A \quad \Rightarrow \quad \Gamma \vdash_{\Sigma} M.N : B \\
  \Gamma \vdash_{A} M : A & \quad A = B \quad \Rightarrow \quad \Gamma \vdash_{\Sigma} M : B
  \end{align*} \]

- The derivation rules for program judgments:
  \[ \begin{align*}
  \Sigma \vdash \emptyset \\
  \Sigma \vdash P \quad \tilde{x} : \tilde{A} \vdash_{\Sigma} q : \bot \quad n : A_1 \rightarrow \ldots \rightarrow A_k \rightarrow \bot \in \Sigma \\
  \Sigma \vdash P \cup \{n.\tilde{x}_{1} \ldots \tilde{x}_{k} \rightarrow q\} \\
  \text{if } n \text{ not defined in } P
  \end{align*} \]

(NB. \( \tilde{x} : \tilde{A} \) denotes \( x_1 : A_1, \ldots, x_k : A_k \))
Call-by-name and call-by-value iteration

To be able to define functions in CC in a generic and safe way, we add for every data-type two program rules for iteration:

- \( \text{ItCBN}_{D \rightarrow B} \) for call-by-name iteration from data-type \( D \) to \( B \), Idea: compute first constructor of the output of type \( B \) and pass to the appropriate continuation for type \( B \).
- \( \text{ItCBV}_{D \rightarrow B} \) for call-by-value iteration from data-type \( D \) to \( B \). Idea: compute input value completely and then pass on to the continuation of type \( B \rightarrow \bot \).
Call-by-name and call-by-value iteration for \( \text{nat} \)

We show the case for \( D = B = \text{nat} \).

Call-by-name:

\[
\begin{align*}
\text{ItCBN}_{\text{nat}\rightarrow\text{nat}}.f_1.f_2.x.c_1.c_2 : \bot
\end{align*}
\]

with associated reduction rules (program rules).

Call-by-value:

\[
\begin{align*}
\text{ItCBV}_{\text{nat}\rightarrow\text{nat}}.f_1.f_2 : \text{nat} \rightarrow \neg\neg\text{nat}
\end{align*}
\]

with associated reduction rules (program rules).
Call-by-name and call-by-value iteration: program rules

Rules for \(\text{ItCBN}_{\text{nat} \rightarrow \text{nat}}\):

\[
\text{ItCBN}.f_1.f_2.x.c_1.c_2 \rightarrow x.(f_1.c_1.c_2).(\text{ItCBN}^{\text{Succ}}.f_1.f_2.c_1.c_2)
\]
\[
\text{ItCBN}^{\text{Succ}}.f_1.f_2.c_1.c_2.x_1 \rightarrow f_2.(\text{ItCBN}.f_1.f_2.x_1).c_1.c_2
\]

Rules for \(\text{ItCBV}_{\text{nat} \rightarrow \text{nat}}\):

\[
\text{ItCBV}.f_1.f_2.x.c \rightarrow x.(f_1.c).(\text{ItCBV}^{\text{Succ},1}.f_1.f_2.c)
\]
\[
\text{ItCBV}^{\text{Succ},1}.f_1.f_2.c.x_1 \rightarrow \text{ItCBV}.f_1.f_2.x_1.(\text{ItCBV}^{\text{Succ},2}.f_1.f_2.c)
\]
\[
\text{ItCBV}^{\text{Succ},2}.f_1.f_2.c.r_1 \rightarrow f_2.r_1.c
\]
Combining call-by-value and call-by-name

A storage operator in CC

\[
\begin{align*}
\text{StoreNat}.n.r & \rightarrow n.(r.\text{Zero}).(A.r) \\
A.r.m & \rightarrow \text{StoreNat}.m.(B.r) \\
B.r.m' & \rightarrow r.(\text{Succ}.m')
\end{align*}
\]

Then StoreNat : nat → ¬¬nat. If \( t \approx \langle p \rangle \), then for any \( r \):

\[
\text{StoreNat}.t.r \rightarrow r.(\text{Succ}^p.\text{Zero})
\]

StoreNat always first evaluates \( t \), before using it. (Mimicking CBV by CBN.)

In the reverse direction: UnstoreNat : ¬¬nat → nat defined by:
given \( f : ¬¬\text{nat}, z : \bot, s : \text{nat} \rightarrow \bot \),

\[
\begin{align*}
\text{UnstoreNat}.f.z.s & \rightarrow f.(\text{UseNat}.z.s) \\
\text{UseNat}.z.s.n & \rightarrow n.z.s
\end{align*}
\]

**Lemma** For all \( n \in \mathbb{N} \),

\[
\text{UnstoreNat}.(\text{StoreNat.}\langle n \rangle) \approx \langle n \rangle.
\]
Results

- Confluence is trivial, because (untyped) CC is deterministic
- Subject reduction holds
- Strong Normalization holds for all CC programs written using
  - constructors of data types and
  - iterators (cbn and cbv) and
  - non-circular well-typed rules
- The SN proof is by
  - translation to a typed $\lambda$-calculus with simple and pos.rec. types and CBN and CBV iterator combinators;
  - proving that this typed $\lambda$-calculus is SN