Proof terms for generalized classical natural deduction

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Standard form for natural deduction rules

$$\frac{\Gamma \vdash A_1 \quad \dots \quad \Gamma \vdash A_n \qquad \Gamma, B_1 \vdash D \quad \dots \quad \Gamma, B_m \vdash D}{\Gamma \vdash D}$$

If the conclusion of a rule is $\Gamma \vdash D$, then the hypotheses of the rule can be of one of two forms:

1. $\Gamma \vdash A$: instead of proving D from Γ, we now need to prove A from Γ. We call A a Lemma.
2. $\Gamma, B \vdash D$: we are given extra data B to prove D from Γ. We call B a Casus.

We don't give the Γ explicitly (it can be retrieved):

$$\frac{\vdash A_1 \quad \dots \quad \vdash A_n \qquad B_1 \vdash D \quad \dots \quad B_m \vdash D}{\vdash D}$$

Natural Deduction rules from truth tables

Let c be an n-ary connective c with truth table t_c.
Each row of t_c gives rise to an elimination rule or an introduction rule for c. (We write $\Phi = c(A_1, \ldots, A_n)$.)

$$\begin{array}{ccc|c} A_1 & \ldots & A_n & \Phi \\ \hline p_1 & \ldots & p_n & 0 \end{array} \quad \mapsto \quad \frac{\vdash \Phi \ldots \vdash A_j \text{ (if } p_j = 1) \ldots A_i \vdash D \text{ (if } p_i = 0) \ldots}{\vdash D}\,\text{el}$$

constructive intro

$$\begin{array}{ccc|c} A_1 & \ldots & A_n & \Phi \\ \hline q_1 & \ldots & q_n & 1 \end{array} \quad \mapsto \quad \frac{\ldots \vdash A_j \text{ (if } q_j = 1) \ldots A_i \vdash \Phi \text{ (if } q_i = 0) \ldots}{\vdash \Phi}\,\text{in}^i$$

classical intro

$$\begin{array}{ccc|c} A_1 & \ldots & A_n & \Phi \\ \hline r_1 & \ldots & r_n & 1 \end{array} \quad \mapsto \quad \frac{\Phi \vdash D \ldots \vdash A_j \text{ (if } r_j = 1) \ldots A_i \vdash D \text{ (if } r_i = 0) \ldots}{\vdash D}\,\text{in}$$

Examples

Rules for $\neg$: 1 elimination rule and 1 introduction rule.

A	$\neg A$
0	1
1	0

Constructive:

$$\frac{\vdash \neg A \quad \vdash A}{\vdash D}\,\neg\text{-el} \qquad \frac{A \vdash \neg A}{\vdash \neg A}\,\neg\text{-in}^{i}$$

Classical:

$$\frac{\vdash \neg A \quad \vdash A}{\vdash D}\,\neg\text{-el} \qquad \frac{\neg A \vdash D \quad A \vdash D}{\vdash D}\,\neg\text{-in}^{c}$$

The rules can be simplified

- From the truth table of a connective c of arity n, we derive 2^n rules.
- These can be optimized to fewer (equivalent) rules. E.g. for $\wedge$ and $\vee$ we then get the well-known derivation rules.

The rules for the classical $\to$ connective

$$\frac{\vdash A \to B \quad \vdash A}{\vdash B} \to\text{-el} \qquad \frac{\vdash B}{\vdash A \to B} \to\text{-in}_1 \qquad \frac{A \to B \vdash D \quad A \vdash D}{\vdash D} \color{red}{\to\text{-in}_2^c}$$

Derivation of Peirce's law:

$$\frac{\dfrac{A \vdash A}{A \vdash ((A \to B) \to A) \to A} \qquad \dfrac{\dfrac{\dfrac{(A \to B) \to A \vdash (A \to B) \to A \quad A \to B \vdash A \to B}{A \to B, (A \to B) \to A \vdash A}}{A \to B, (A \to B) \to A \vdash ((A \to B) \to A) \to A}}{A \to B \vdash ((A \to B) \to A) \to A}}{\vdash ((A \to B) \to A) \to A} \to\text{-in}_2^c$$

Some properties

- For monotone connectives, the constructive and classical rules are equivalent. (E.g. $\wedge$, $\vee$)
- For the non-monotonic connectives $\to$ and $\neg$, the classical intro rule for $\to$ implies the classical intro rule for $\neg$ (and vice versa).
- This holds in general: one classical intro rule for a non-monotonic connective makes all connectives classical.

(NB. c is monotonic if the truth table of c, $t_c : \{0,1\}^n \to \{0,1\}$ is a monotonic function w.r.t. the ordering induced by $0 \leq 1$.)

$$t ::= x \mid (\lambda y : A.t) \star_r \{\overline{t} \, ; \overline{\lambda x : A.t}\} \mid t \cdot_r [\overline{t} \, ; \overline{\lambda x : A.t}]$$

where x ranges over variables and r ranges over the rules of all the connectives.

The terms are typed using the following derivation rules.

$$\frac{}{\Gamma \vdash x_i : A_i} \text{ if } x_i : A_i \in \Gamma$$

$$\frac{z : \Phi \vdash t : D \quad \ldots \vdash p_i : A_i \ldots \quad \ldots y_j : A_j \vdash q_j : D \ldots}{\vdash (\lambda z : \Phi.t) \star_r \{\overline{p} \, ; \overline{\lambda y : A.q}\} : D} \text{ in}$$

$$\frac{\vdash t : \Phi \quad \ldots \vdash p_k : A_k \ldots \quad \ldots y_\ell : A_\ell \vdash q_\ell : D}{\vdash t \cdot_r [\overline{p} \, ; \overline{\lambda y : A.q}] : D} \text{ el}$$

Reduction for proof terms in classical logic

- First perform permutation reductions.
- Then we perform detour reductions.

This is similar to the constructive case, except for now

- a term is in permutation normal form if all lemmas are variables,
- a detour is an elimination of Φ followed by an introduction of Φ.

NB: in constructive logic, a "detour" is an introduction directly followed by an elimination. Here it is the other way around, and the introduction need not follow the elimination directly.

We obtain a deduction in permutation normal form by moving elimination or introduction rules that have a non-trivial lemma upwards, until all lemmas become trivial: the proof-terms are variables. (This only works for the classical case!)

A detour is a pattern of the following shape

$$(\lambda x : \Phi. \ldots (x \cdot [\overline{v}\,;\, \overline{\lambda w : A.s}]) \ldots) \star \{\overline{z}\,;\, \overline{\lambda y : A.q}\}$$

that is, an elimination of $\Phi = c(A_1, \ldots, A_n)$ followed by an introduction of Φ, with an arbitrary number of steps in between.

- For terms in permutation normal form, detours can be eliminated,
- One obtains a term in normal form which satisfies the sub-formula property.

Notes to the pattern of a detour:

- the indicated occurrence need not be the only occurrence of x
- variable x may not occur at all; that is the simplest situation.

Eliminating detours is done by the following reduction steps:

- $(\lambda x : \Phi. \ldots (x \cdot [\overline{v} \,;\, \overline{\lambda w : A.s}]) \ldots) \star \{\overline{z} \,;\, \overline{\lambda y : A.q}\} \longrightarrow_a$
$$(\lambda x : \Phi. \ldots (s_\ell[w_\ell := z_i]) \ldots) \star \{\overline{z} \,;\, \overline{\lambda y : A.q}\}$$
if $i = \ell\,(A_i = A_\ell)$ is a "matching case" for the subformulas of Φ.

- $(\lambda x : \Phi. \ldots (x \cdot [\overline{v} \,;\, \overline{\lambda w : A.s}]) \ldots) \star \{\overline{z} \,;\, \overline{\lambda y : A.q}\} \longrightarrow_a$
$$(\lambda x : \Phi. \ldots (q_j[y_j := v_k]) \ldots) \star \{\overline{z} \,;\, \overline{\lambda y : A.q}\}$$
if $j = k\,(A_j = A_k)$ is a "matching case" for the subformulas of Φ.

- $(\lambda x : \Phi.t) \star \{\overline{z} \,;\, \overline{\lambda y : A.q}\} \longrightarrow_a t \qquad$ if $x \notin \mathrm{FV}(t)$.

Tonny Hurkens has given a proof that this normalizes

Permutation conversion for the $\to$-case

The rules for implication are as follows.

$$\frac{t : A \to B \quad a : A \quad y : B \vdash q : D}{\vdash t \cdot [a\, ;\, \lambda y.q] : D}\ \text{el-f} \qquad \frac{t : A \to B \quad a : A}{\vdash t \cdot [a\, ;\, -] : B}\ \text{el-s}$$

$$\frac{x : A \to B \vdash t : D \quad y : A \vdash q : D}{\vdash (\lambda x.t) \star \{\ ;\, \lambda y.q\} : D}\ \text{in}^c \qquad \frac{x : A \to B \vdash t : D \quad b : B}{\vdash (\lambda x.t) \star \{b\, ;\ \} : D}\ \text{in}_1$$

el-s and el-f are equivalent: $t \cdot [a\, ;\, -] := t \cdot [a\, ;\, \lambda y.y]$, but for permutation reduction, el-f is essential.

Example (I)

$$\dfrac{t : B \to C \qquad \dfrac{s : A \to B \quad a : A}{s \cdot [a\,;\,-] : B}\,\text{el-s}}{t \cdot [s \cdot [a\,;\,-]\,;\,-] : C}\,\text{el-s}$$

This proof is not in permutation normal form.

It reduces to

$$\dfrac{s : A \to B \quad a : A \qquad \dfrac{t : B \to C \quad (y : B)^1}{t \cdot [y\,;\,-] : C}\,\text{el-s}}{s \cdot [a\,;\,\lambda y.t \cdot [y\,;\,-]] : C}\,\text{el-f}_{(1)}$$

Example (II)

$$\dfrac{\dfrac{t : A \to B \to C \quad a : A}{t \cdot [a \,;\, -] : B \to C}\,\text{el-s} \qquad b : B}{t \cdot [a \,;\, -] \cdot [b \,;\, -] : C}\,\text{el-s}$$

This proof is not in permutation normal form.

It reduces to

$$\dfrac{t : A \to B \to C \quad a : A \qquad \dfrac{(y : B \to C)^1 \quad b : B}{y \cdot [b \,;\, -] : C}\,\text{el-s}}{t \cdot [a \,;\, \lambda y.y \cdot [b \,;\, -]] : C}\,\text{el-f}_{(1)}$$

Conclusions

- Simple general way to derive classical deduction rules for (new) connectives.
- One can study connectives "in isolation". (Without other connectives.)
- General Curry-Howard proofs-as-terms interpretation.
- General definitions of detour conversion and permutation conversion.
- General Normalization proof.

Questions?