

Proof terms for generalized classical natural deduction

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Standard form for natural deduction rules

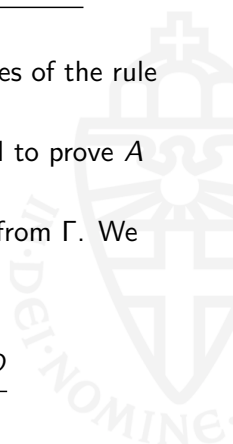
$$\frac{\Gamma \vdash A_1 \quad \dots \quad \Gamma \vdash A_n \quad \Gamma, B_1 \vdash D \quad \dots \quad \Gamma, B_m \vdash D}{\Gamma \vdash D}$$

If the conclusion of a rule is $\Gamma \vdash D$, then the hypotheses of the rule can be of one of two forms:

- 1 $\Gamma \vdash A$: instead of proving D from Γ , we now need to prove A from Γ . We call A a **Lemma**.
- 2 $\Gamma, B \vdash D$: we are given extra data B to prove D from Γ . We call B a **Casus**.

We don't give the Γ explicitly (it can be retrieved):

$$\frac{\vdash A_1 \quad \dots \quad \vdash A_n \quad B_1 \vdash D \quad \dots \quad B_m \vdash D}{\vdash D}$$



Natural Deduction rules from truth tables

Let c be an n -ary connective c with truth table t_c .

Each row of t_c gives rise to an elimination rule or an introduction rule for c . (We write $\Phi = c(A_1, \dots, A_n)$.)

$$\frac{A_1 \quad \dots \quad A_n \mid \Phi}{p_1 \quad \dots \quad p_n \mid 0} \mapsto \frac{\vdash \Phi \dots \vdash A_j \text{ (if } p_j = 1) \dots A_i \vdash D \text{ (if } p_i = 0) \dots}{\vdash D} \text{el}$$

constructive intro

$$\frac{A_1 \quad \dots \quad A_n \mid \Phi}{q_1 \quad \dots \quad q_n \mid 1} \mapsto \frac{\dots \vdash A_j \text{ (if } q_j = 1) \dots A_i \vdash \Phi \text{ (if } q_i = 0) \dots}{\vdash \Phi} \text{in}^i$$

classical intro

$$\frac{A_1 \quad \dots \quad A_n \mid \Phi}{r_1 \quad \dots \quad r_n \mid 1} \mapsto \frac{\Phi \vdash D \dots \vdash A_j \text{ (if } r_j = 1) \dots A_i \vdash D \text{ (if } r_i = 0) \dots}{\vdash D} \text{in}$$

Examples

Rules for \neg : 1 elimination rule and 1 introduction rule.

A	$\neg A$
0	1
1	0

Constructive:

$$\frac{\vdash \neg A \quad \vdash A}{\vdash D} \neg\text{-el} \qquad \frac{A \vdash \neg A}{\vdash \neg A} \neg\text{-in}^i$$

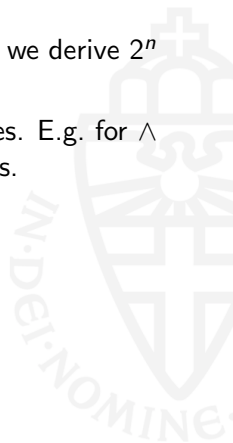
Classical:

$$\frac{\vdash \neg A \quad \vdash A}{\vdash D} \neg\text{-el} \qquad \frac{\neg A \vdash D \quad A \vdash D}{\vdash D} \neg\text{-in}^c$$



The rules can be simplified

- From the truth table of a connective c of arity n , we derive 2^n rules.
- These can be optimized to fewer (equivalent) rules. E.g. for \wedge and \vee we then get the well-known derivation rules.



The rules for the classical \rightarrow connective

$$\frac{\vdash A \rightarrow B \quad \vdash A}{\vdash B} \rightarrow\text{-el} \qquad \frac{\vdash B}{\vdash A \rightarrow B} \rightarrow\text{-in}_1 \qquad \frac{A \rightarrow B \vdash D \quad A \vdash D}{\vdash D} \rightarrow\text{-in}_2^C$$

Derivation of Peirce's law:

$$\frac{\frac{\frac{A \vdash A}{A \vdash ((A \rightarrow B) \rightarrow A) \rightarrow A}}{A \rightarrow B, (A \rightarrow B) \rightarrow A \vdash ((A \rightarrow B) \rightarrow A) \rightarrow A} \rightarrow\text{-in}_2^C}{\vdash ((A \rightarrow B) \rightarrow A) \rightarrow A} \rightarrow\text{-in}_2^C$$

Some properties

- For **monotone** connectives, the constructive and classical rules are equivalent. (E.g. \wedge , \vee)
- For the **non-monotonic** connectives \rightarrow and \neg , the **classical intro rule** for \rightarrow implies the **classical intro rule** for \neg (and vice versa).
- This holds in general: one classical intro rule for a non-monotonic connective makes all connectives classical.

(NB. c is monotonic if the truth table of c , $t_c : \{0, 1\}^n \rightarrow \{0, 1\}$ is a monotonic function w.r.t. the ordering induced by $0 \leq 1$.)

Curry-Howard proofs-as-terms for classical logic

$$t ::= x \mid (\lambda y : A.t) \star_r \{\bar{t}; \overline{\lambda x : A.t}\} \mid t \cdot_r [\bar{t}; \overline{\lambda x : A.t}]$$

where x ranges over variables and r ranges over the rules of all the connectives.

The terms are typed using the following derivation rules.

$$\frac{}{\Gamma \vdash x_i : A_i} \text{ if } x_i : A_i \in \Gamma$$
$$\frac{z : \Phi \vdash t : D \quad \dots \vdash p_i : A_i \dots \quad \dots y_j : A_j \vdash q_j : D \dots}{\vdash (\lambda z : \Phi.t) \star_r \{\bar{p}; \overline{\lambda y : A.q}\} : D} \text{ in}$$
$$\frac{\vdash t : \Phi \quad \dots \vdash p_k : A_k \dots \quad \dots y_\ell : A_\ell \vdash q_\ell : D}{\vdash t \cdot_r [\bar{p}; \overline{\lambda y : A.q}] : D} \text{ el}$$

Reduction for proof terms in classical logic

- First perform **permutation reductions**.
- Then we perform **detour reductions**.

This is similar to the constructive case, except for now

- a term is in **permutation normal form** if all lemmas are variables,
- a **detour** is an elimination of Φ followed by an introduction of Φ .

NB: in constructive logic, a “detour” is an introduction **directly followed** by an elimination. Here it is the other way around, and the introduction need not follow the elimination directly.

We obtain a deduction in **permutation normal** form by moving elimination or introduction rules that have a non-trivial lemma upwards, until all lemmas become trivial: the proof-terms are variables. (This only works for the classical case!)

Detours for proof terms in classical logic

A **detour** is a pattern of the following shape

$$(\lambda x : \Phi \dots (x \cdot [\bar{v} ; \overline{\lambda w : A.s}]) \dots) \star \{\bar{z} ; \overline{\lambda y : A.q}\}$$

that is, an elimination of $\Phi = c(A_1, \dots, A_n)$ followed by an introduction of Φ , with an arbitrary number of steps in between.

- For terms in permutation normal form, detours can be eliminated,
- One obtains a term in normal form which satisfies the **sub-formula property**.

Notes to the pattern of a detour:

- the indicated occurrence need not be the only occurrence of x
- variable x may not occur at all; that is the simplest situation.

Eliminating detours is done by the following reduction steps:

- $(\lambda x : \Phi \dots (x \cdot [\bar{v} ; \overline{\lambda w : A.s}]) \dots) \star \{\bar{z} ; \overline{\lambda y : A.q}\} \longrightarrow_a$
 $(\lambda x : \Phi \dots (s_\ell[w_\ell := z_\ell]) \dots) \star \{\bar{z} ; \overline{\lambda y : A.q}\}$
if $i = \ell$ ($A_i = A_\ell$) is a “matching case” for the subformulas of Φ .
- $(\lambda x : \Phi \dots (x \cdot [\bar{v} ; \overline{\lambda w : A.s}]) \dots) \star \{\bar{z} ; \overline{\lambda y : A.q}\} \longrightarrow_a$
 $(\lambda x : \Phi \dots (q_j[y_j := v_k]) \dots) \star \{\bar{z} ; \overline{\lambda y : A.q}\}$
if $j = k$ ($A_j = A_k$) is a “matching case” for the subformulas of Φ .
- $(\lambda x : \Phi.t) \star \{\bar{z} ; \overline{\lambda y : A.q}\} \longrightarrow_a t$ if $x \notin \text{FV}(t)$.

Tonny Hurkens has given a proof that this normalizes

Permutation conversion for the \rightarrow -case

The rules for implication are as follows.

$$\frac{t : A \rightarrow B \quad a : A \quad y : B \vdash q : D}{\vdash t \cdot [a ; \lambda y. q] : D} \text{el-f}$$

$$\frac{x : A \rightarrow B \vdash t : D \quad y : A \vdash q : D}{\vdash (\lambda x. t) \star \{ ; \lambda y. q \} : D} \text{in}^c$$

$$\frac{t : A \rightarrow B \quad a : A}{\vdash t \cdot [a ; -] : B} \text{el-s}$$

$$\frac{x : A \rightarrow B \vdash t : D \quad b : B}{\vdash (\lambda x. t) \star \{ b ; \} : D} \text{in}_1$$

el-s and el-f are equivalent: $t \cdot [a ; -] := t \cdot [a ; \lambda y. y]$, but for permutation reduction, el-f is essential.

Example (I)

$$\frac{t : B \rightarrow C \quad \frac{s : A \rightarrow B \quad a : A}{s \cdot [a ; -] : B} \text{el-s}}{t \cdot [s \cdot [a ; -] ; -] : C} \text{el-s}$$

This proof is not in permutation normal form.

It reduces to

$$\frac{s : A \rightarrow B \quad a : A \quad \frac{t : B \rightarrow C \quad (y : B)^1}{t \cdot [y ; -] : C} \text{el-s}}{s \cdot [a ; \lambda y. t \cdot [y ; -]] : C} \text{el-f}_{(1)}$$



Example (II)

$$\frac{\frac{t : A \rightarrow B \rightarrow C \quad a : A}{t \cdot [a ; -] : B \rightarrow C} \text{el-s} \quad b : B}{t \cdot [a ; -] \cdot [b ; -] : C} \text{el-s}$$

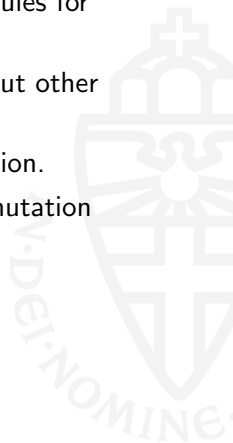
This proof is not in permutation normal form.

It reduces to

$$\frac{t : A \rightarrow B \rightarrow C \quad a : A \quad \frac{(y : B \rightarrow C)^1 \quad b : B}{y \cdot [b ; -] : C} \text{el-s}}{t \cdot [a ; \lambda y. y \cdot [b ; -]] : C} \text{el-f}_{(1)}$$



- Simple general way to derive classical deduction rules for (new) connectives.
- One can study connectives “in isolation”. (Without other connectives.)
- General Curry-Howard proofs-as-terms interpretation.
- General definitions of detour conversion and permutation conversion.
- General Normalization proof.



Questions?

