Pinchuk's 2-dimensional example paired to a cubic linear 1999-dimensional map

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Abstract

Back in 1994 Sergey Pinchuk has presented pairs of real polynomials in two variables that do have a nowhere vanishing Jacobian determinant but are not one-to-one. In this paper we examine a specific example $F := (P, Q)$ where $\deg(P) = 10$ and $\deg(Q) = 25$. We use a slightly modified version of the technique described in [1] by Bass, Connell and Wright to reduce this map to a cubic homogeneous map. Furthermore we transform this map into a cubic linear (or Drużkowski map) using the pairing-technique by Gorni and Zampieri (see [3]). The result is an example in dimension 1999.

1 Introduction

The following conjecture is known as the Real Jacobian Conjecture:

Conjecture 1.1 *Let* $F : \mathbb{R}^n \to \mathbb{R}^n$ *be a polynomial map such that* $\det(JF(x)) \neq 0$ *for all* $x \in \mathbb{R}^n$. *Then* F *is injective.*

This conjecture implies the Jacobian Conjecture. Let $F : \mathbb{C}^n \to \mathbb{C}^n$ be a polynomial map with $\det(JF) \in \mathbb{C}^*$. Define $\tilde{F} : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ by splitting the real and the imaginair part: $\tilde{F} := (\Re F_1, \Im F_1, \ldots, \Re F_n, \Im F_n)$. Then $\det(J\tilde{F}) = |\det(JF)|^2 \in \mathbb{R}^*$. So if the Real Jacobian Conjecture holds, $\tilde{F}$ and hence F is injective and invertible (see [2]).

However, in 1994 Pinchuk came up with the following counterexample in dimension 2! (See [5].) Define in $\mathbb{R}[y_1, y_2]$

$$\begin{aligned} t &:= y_1 y_2 - 1 \\ h &:= t(y_1 t + 1) \end{aligned}$$

$$
\begin{aligned}
f &:= \frac{h+1}{y_1}(y_1 t + 1)^2 \\
u &:= 170fh + 91h^2 + 195fh^2 + 69h^3 + 75h^3 f + \frac{75}{4}h^4 \\
P &:= f + h \\
Q &:= -t^2 - 6th(h+1) - u
\end{aligned}
$$

then

$$F := (P, Q)$$

is a counterexample to the Real Jacobian Conjecture. (One easily verifies that $\det(JF) = t^2 + (t + f(13 + 15h))^2 + f^2$ and this term is > 0 on $\mathbb{R}^2$ since it can only be zero if both t and f are zero. But if $t = 0$ then $f = \frac{1}{y_1} \neq 0$. And $F(1, 0) = F(-1, -2)$ which means that F is not injective.)

During the May 1997 conference honouring the mathematical work of Gary Meisters held in Lincoln, Nebraska, this example was discussed several times. One of the speakers, Parthasarathy, explicitly asked what this example would look like if it was transformed into a Drużkowski form using the methods of [1] and [3]. This question was the starting point for this paper. After the conference the author tried to compute the answer to this question. However he failed, due to computational limitations. Fortunately, the process was completed succesfully far enough to provide the dimension in which a paired cubic linear example exists: in dimension 2033. The way the author came to this solution was described in an unpublished draft: [4].

Now in October 1999 it turns out that this draft has been crucial to find the answer to Parthasarathy's question of 1997. Because the result in the draft was not very impressive, the author didn't bother to spread it around his colleagues. Only Parthasarathy and the author's supervisor van den Essen knew about it. Therefore the author was quite surprised when he received an e-mail in June 1999 by Michael Kinyon, Indiana University South Bend, in which Kinyon put some questions concerning that project. Inspired by this attention, the author decided to get back to this problem. And after some weeks of labour, he found out that the failure of the project in 1997 was not really due to computational limits of the machine[1] he used, but more to the specific commands he used in Maple to get the desired cubic linear example. This paper is a reflection of the new project.

[1]Computations were done on an eight-processor machine with 170MHz Ultra Sparc processors and 1 Gb of RAM.

2 Reduction to a polynomial map of degree three

The GZ algorithm as presented by Gorni and Zampieri in [3] computes for a given cubic homogeneous map $f : \mathbb{C}^n \to \mathbb{C}^n$ a cubic linear map $F : \mathbb{C}^N \to \mathbb{C}^N$ where $N > n$. In particular one needs a cubic homogeneous map as the input for this algorithm. And as stated earlier Pinchuk's example is of (total) degree 10 and 25 in its two components. So reducing to a cubic homogeneous map would be the first task to do.

Basically this is done by the algorithm presented in [1]. However, the original algorithm operates on monomials: for each monomial of degree > 3 it introduces new variables to reduce this monomial. But running this standard algorithm didn't give very promising results: it 'reduced' Pinchuk's example to a map $F' : \mathbb{C}^{357} \to \mathbb{C}^{357}$. Using the standard algorithm this would imply a cubic homogenous map $F'' : \mathbb{C}^{715} \to \mathbb{C}^{715}$. And afterwards this map F'' should be transformed to a Drużkowski map in dimension N where $N > 715$ (and in particular N can be much bigger than 715).

Hence in order to reduce the number of extra variables, we applied the Bass-Connel-Wright algorithm (BCW algorithm) to *poly*nomials instead of *mo*nomials. One of the arguments in favour of this choice, is the fact that Pinchuk's example is already built by the polynomials f, h and t.

Before we present some actual results we need to make some comments. The ideal result would be obtained by reducing F to a mapping where f appears only linearly. Then substituting f as a function in h, t and y_1 and reducing to a mapping where the h appears only linearly. At this point substituting $h = t(y_1 t + 1)$ and reducing to a mapping linear in t. Finally substituting $t = y_1 y_2 - 1$ and reducing to a mapping of total degree 3 in y_1, y_2 and the extra variables added. However, there is a small problem with this approach: $f = \frac{h+1}{y_1}(y_1 t + 1)^2$ so we must be very careful in order to get a *polynomial* mapping at the end. In particular this means that for the reduction of this map, one must hold the $\frac{h+1}{y_1}$ together. However, this is not the normal result of the algorithm, whereas one only looks at the terms of highest degree, so the 'h' and the '1' are likely to be split up, which means that even after substituting h and t we cannot get rid of the $\frac{1}{y_1}$ term. In particular this would probably leave us with a rational function instead of a polynomial map.

In order to solve this problem we introduced $g := \frac{h+1}{y_1}$ and changed tactics a bit. This is the approach we used:

1. Note that

$$\deg_{y_1,y_2}(t) \quad = \quad 2$$

$$\begin{aligned}\deg_{y_1,y_2}(g) &= 4\\ \deg_{y_1,y_2}(h) &= 5\\ \deg_{y_1,y_2}(f) &= 10\end{aligned}$$

We start by wiping out the monomials with highest degree in y_1 and y_2.

2. Note that $\deg_f(F) = 1$.
3. Substitute $f = g(y_1t + 1)^2$.
4. Add new variables to obtain that $\deg_h(F) = 1$.
5. Add new variables to obtain that $\deg_{h,g}(F) = 1$.
6. Add new variables to obtain that $\deg_{g,t}(F) = 1$.
7. Substitute $h = t(y_1t + 1)$.
8. Add new variables to obtain that $\deg_{g,t}(F) = 1$.
9. Substitute $g = \frac{h+1}{y_1}$, $h = t(y_1t + 1)$ and $t = y_1y_2 - 1$.
10. Add new variables to obtain that $\deg_{y_1,y_2,x_1,\ldots,x_m}(F) = 3$, where $x_1, \ldots, x_m$ are the new variables added in the complete process.

Normally, the BCW algorithm assumes $F(0, 0) = (0, 0)$ but here $F(0, 0) = (0, \frac{33}{4})$. Since it is not clear what happens to the constant part if one views F as a map in f, h and t, we take care of this constant part only after adding all extra variables.

At each step the reduction consists of finding suitable L and R such that $L \circ F^{[m]} \circ R$ eliminates one of the higher degree terms in some component. The structure of these maps is always the same and in particular both L and R are almost equal to the identity map. Assume we are reducing a term S of the i-th component of $F : \mathbb{C}^n \to \mathbb{C}^n$. Let $S = c_iS_1S_2$ where c_i is a constant and both S_1 and S_2 have degree at least two. At this point we have three possibilities:

- There exist k and l such that the terms $x_k + S_1$ and $x_l + S_2$ are components of F. Hence we don't have to add new variables at all to wipe out S. Just put

$$\begin{aligned}L_j &= x_j \quad j \in \{1, \ldots, \hat{i}, \ldots, n\}\\ L_i &= x_i - c_ix_kx_l\\ R_j &= x_j \quad j \in \{1, \ldots, n\}\end{aligned}$$

- There exists k such that $x_k + S_1$ is a component of F but there is no l such that $x_l + S_2$ is a component of F. (Possibly after swapping S_1 and S_2.)

$$\begin{aligned} L_j &= x_j \quad j \in \{1, \ldots, \hat{\imath}, \ldots, n+1\} \\ L_i &= x_i - c_i x_k x_{n+1} \\ R_j &= x_j \quad j \in \{1, \ldots, n\} \\ R_{n+1} &= x_{n+1} + S_2 \end{aligned}$$

- There exists no k such that $x_k + S_1$ or $x_k + S_2$ are components of F. Put

$$\begin{aligned} L_j &= x_j \quad j \in \{1, \ldots, \hat{\imath}, \ldots, n+2\} \\ L_i &= x_i - c_i x_k x_{n+1} \\ R_j &= x_j \quad j \in \{1, \ldots, n\} \\ R_{n+1} &= x_{n+1} + S_1 \\ R_{n+2} &= x_{n+2} + S_2 \end{aligned}$$

Hence it suffices to present L_i, R_{n+1} and R_{n+2} (if the last two exist) to give a full description of L and R.

In order to be able to check automatically whether S_i is a known factor, we make sure that the leading coefficent of S_i equals 1.

Below we present the actual list of L_i's and R_i's. It is compiled out of blocks. These blocks represent certain parts of the strategy described above:

- 1–4: reduce until $\deg_h(F) = 1$ (step 4).
- 5–7: reduce until $\deg_{h,g}(F) = 1$ (step 5).
- 8–25: reduce until $\deg_{g,t}(F) = 1$ (step 6).
- 26–34: reduce until $\deg_{g,t}(F) = 1$ (step 8).
- 35–107: reduce until $\deg_{y_1,y_2,x_1,\ldots,x_{99}}(F) = 1$ (step 10).

If an index appears twice in the list this means that R_i has two nontrivial entries as in the third case explained before. The entries are exported by Maple to the LaTeX format, hence typing errors do not occur.

	L	R

	L	R
1	$y_2 + \frac{75}{4} x_1{}^2$	$x_1 + h^2$
2	$y_2 - x_1 x_2$	$x_2 - 3 (50 g t y_1 + 25 g + 23 + 25 g y_1{}^2 t^2) h$
3	$y_2 - x_3 x_4$	$x_3 - \frac{1}{2} (390 g y_1{}^2 t^2 + 2 x_2 + 12 t + 182 + 390 g - 75 x_1 + 780 g t y_1) h$
3		$x_4 + h$
4	$x_1 - x_4{}^2$	
5	$y_2 - x_4 x_5$	$x_5 + 5 (y_1 t + 1)^2 (39 x_4 - 34 + 15 x_1) g$
6	$x_2 - x_4 x_6$	$x_6 - 75 (y_1 t + 1)^2 g$
7	$x_3 - x_4 x_7$	$x_7 - 195 (y_1 t + 1)^2 g$
8	$y_1 - x_8 x_9$	$x_8 + y_1{}^2 t^2$
8		$x_9 + g$
9	$y_2 - x_9 x_{10}$	$x_{10} - 5 y_1{}^2 x_4 (39 x_4 - 34 + 15 x_1) t^2$
10	$x_2 - 75 x_9 x_{11}$	$x_{11} + y_1{}^2 t^2 x_4$
11	$x_3 - 195 x_9 x_{11}$	
12	$x_5 - x_9 x_{12}$	$x_{12} + 5 y_1{}^2 (39 x_4 - 34 + 15 x_1) t^2$
13	$x_6 + 75 x_8 x_9$	
14	$x_7 + 195 x_8 x_9$	
15	$y_1 + x_{13} x_{14}$	$x_{13} + y_1 t$
15		$x_{14} + t x_9 y_1 - 2 g$
16	$y_2 - x_{16} x_{15}$	$x_{15} + t$
16		$x_{16} - 390 g y_1 x_4{}^2 + 340 g y_1 x_4 - 150 x_4 g x_1 y_1 + 75 t x_9 x_4 x_1 y_1{}^2 - 170 t x_9 y_1{}^2 x_4 + 195 t x_9 y_1{}^2 x_4{}^2 - t$
17	$x_2 + 75 x_{17} x_{18}$	$x_{17} + t y_1 x_4$
17		$x_{18} + t x_9 y_1 - 2 g$
18	$x_3 + 195 x_{17} x_{18}$	
19	$x_5 + 5 x_{19} x_{18}$	$x_{19} + t y_1 (39 x_4 - 34 + 15 x_1)$
20	$x_6 - 75 x_{13} x_{18}$	
21	$x_7 - 195 x_{13} x_{18}$	
22	$x_8 - x_{13}{}^2$	
23	$x_{10} + 5 x_{17} x_{19}$	
24	$x_{11} - x_{17} x_{13}$	
25	$x_{12} - 5 x_{19} x_{13}$	
26	$y_2 - x_{20} x_{21}$	$x_{20} + t^2$
26		$x_{21} + 6 y_1 (x_4 - 1) t$
27	$x_3 + 6 x_{13} x_{20}$	
28	$y_1 - x_{15} x_{13}$	

	L	R
29	$y_2 + \frac{1}{2}\, x_{15}\, x_{22}$	$x_{22} + (75\, x_4\, x_1\, y_1 + 2\, x_{21} - 182\, x_4\, y_1 - 138\, x_1\, y_1 - 12\, x_4 + 12 + 2\, x_5\, y_1 - 2\, x_2\, x_4\, y_1 + 2\, x_3\, y_1)t$
30	$x_1 + 2\, x_{13}\, x_{23}$	$x_{23} + x_4\, t$
31	$x_2 - x_{15}\, x_{24}$	$x_{24} - y_1\, (69 + x_6)\, t$
32	$x_3 - \frac{1}{2}\, x_{15}\, x_{25}$	$x_{25} + (75\, x_1\, y_1 - 2\, x_7\, y_1 - 182\, y_1 - 2\, x_2\, y_1 + 12\, x_{13} - 12)\, t$
33	$x_4 - x_{15}\, x_{13}$	
34	$x_{20} - {x_{15}}^2$	
35	$y_2 + 195\, x_{26}\, x_{27}$	$x_{26} + {y_1}^3$
35		$x_{27} + \frac{1}{13}\, y_2\, x_{15}\, x_4\, (5\, x_1 + 13\, x_4)\, (-2\, y_2 + x_9)$
36	$y_2 - 170\, x_{26}\, x_{28}$	$x_{28} + x_4\, x_{15}\, y_2\, (-2\, y_2 + x_9)$
37	$x_{16} - 195\, x_{26}\, x_{29}$	$x_{29} + \frac{1}{13}\, x_4\, y_2\, (5\, x_1 + 13\, x_4)\, (-2\, y_2 + x_9)$
38	$y_2 + 195\, x_{30}\, x_{31}$	$x_{30} + {y_1}^2$
38		$x_{31} + \frac{1}{13}\, x_4\, {y_2}^2\, (5\, x_1 + 13\, x_4)$
39	$x_{16} + 170\, x_{26}\, x_{32}$	$x_{32} + x_4\, y_2\, (-2\, y_2 + x_9)$
40	$y_2 - 195\, x_{33}\, x_{34}$	$x_{33} + x_{15}\, x_4$
40		$x_{34} - 2\, x_4\, x_{26}\, {y_2}^2 + x_9\, {y_1}^2\, x_4 + x_{26}\, y_2\, x_4\, x_9 - 4\, x_4\, {y_1}^2\, y_2 - \frac{1}{195}\, x_2\, {y_1}^2\, y_2 + \frac{5}{13}\, x_9\, x_1\, {y_1}^2 - \frac{10}{13}\, x_{26}\, x_1\, {y_2}^2 - \frac{35}{26}\, x_1\, {y_1}^2\, y_2 + \frac{5}{13}\, x_{26}\, x_9\, x_1\, y_2$
41	$y_1 - x_{30}\, x_{35}$	$x_{35} - y_2\, (y_2\, x_8 + 2\, x_{13}\, y_2 - x_9\, x_{13})$
42	$y_2 + 195\, x_{30}\, x_{36}$	$x_{36} + \frac{1}{39}\, x_9\, (34\, x_{15}\, x_4 + 15\, x_1\, x_{33} + 39\, x_4\, x_{33})$
43	$x_2 + 75\, x_{30}\, x_{37}$	$x_{37} - \frac{1}{75}\, y_2\, (-75\, x_{11}\, y_2 - 150\, x_{17}\, y_2 + 75\, x_4\, y_2 + x_6\, x_{15} + 75\, x_{18}\, x_4 + 75\, x_9\, x_{17})$
44	$x_3 + 195\, x_{38}\, x_{30}$	$x_{38} + \frac{1}{390} y_2(-390\, x_4\, y_2 + 780\, x_{17}\, y_2 + 390\, x_{11}\, y_2 - 2\, x_2\, x_{15} - 2\, x_7\, x_{15} + 75\, x_1\, x_{15} - 390\, x_9\, x_{17} - 390\, x_{18}\, x_4)$
45	$x_5 + x_{30}\, x_{39}$	$x_{39} - y_2\, (195\, x_4\, y_2 - 10\, x_{19}\, y_2 + 75\, y_2\, x_1 - x_{12}\, y_2 + 5\, x_9\, x_{19} + 195\, x_{18}\, x_4 + 75\, x_1\, x_{18})$
46	$x_6 + 75\, x_{30}\, x_{35}$	
47	$x_7 + 195\, x_{30}\, x_{35}$	
48	$x_{10} - 195\, x_{30}\, x_{40}$	$x_{40} + \frac{1}{39}\, y_2\, (15\, x_{17}\, x_1 + 39\, x_{17}\, x_4 + x_4\, x_{19})$
49	$x_{11} + x_{30}\, x_{41}$	$x_{41} + x_4\, x_{13}\, y_2$
50	$x_{12} + 195\, x_{30}\, x_{42}$	$x_{42} + \frac{1}{13}\, x_{13}\, y_2\, (5\, x_1 + 13\, x_4)$
51	$x_{16} + 195\, x_{44}\, x_{43}$	$x_{43} + \frac{1}{13}\, x_4\, (5\, x_1 + 13\, x_4)$
51		$x_{44} - 4\, {y_1}^2\, y_2 + x_9\, x_{26}\, y_2 - 2\, x_{26}\, {y_2}^2 + x_9\, {y_1}^2$
52	$x_{22} + 2\, x_{30}\, x_{45}$	$x_{45} - \frac{1}{2}\, x_4\, y_2\, (-2\, x_2 + 75\, x_1)$
53	$x_{27} - x_{46}\, x_{47}$	$x_{46} + x_4\, y_2$

	L	R
53		$x_{47} + \frac{1}{13}\,x_{15}\,(5\,x_1 + 13\,x_4)\,(-2\,y_2 + x_9)$
54	$y_2 - 195\,x_{48}\,x_{49}$	$x_{48} + {y_2}^2$
54		$x_{49} + \frac{1}{13}\,x_{30}\,x_4\,(5\,x_1 + 13\,x_4)$
55	$y_2 + 195\,x_{50}\,x_{51}$	$x_{50} + y_2\,x_{26}$
55		$x_{51} + \frac{1}{39}\,(34\,x_{15}\,x_4 + 15\,x_1\,x_{33} + 39\,x_4\,x_{33})\,(-2\,y_2 + x_9)$
56	$y_2 + 6\,x_{52}\,x_{53}$	$x_{52} + y_1\,y_2$
56		$x_{53} - \frac{85}{3}\,x_4\,y_1\,y_2 + \frac{1}{6}\,x_{10}\,y_1\,y_2 - \frac{1}{6}\,y_1\,x_2\,x_{33} - 130\,y_1\,x_4\,x_{33} - \frac{1}{6}\,x_{15}\,x_5\,y_1 - \frac{175}{4}\,y_1\,x_1\,x_{33} + \frac{23}{2}\,x_{15}\,x_1\,y_1 - \frac{1}{6}\,x_{15}\,x_3\,y_1 + x_{20}\,y_1\,x_4 - \frac{589}{6}\,y_1\,x_4\,x_{15} - 25\,x_{15}\,x_1\,x_4 - 65\,x_{15}\,{x_4}^2$
57	$y_1 - x_{30}\,x_{54}$	$x_{54} + y_2\,(y_2 + x_{14} - x_{15})$
58	$y_2 + 340\,x_{33}\,x_{55}$	$x_{55} + \frac{39}{34}\,x_4\,x_{52} + \frac{15}{34}\,x_{52}\,x_1 + y_2\,x_{50}$
59	$x_1 - 2\,x_{52}\,x_{56}$	$x_{56} + x_{23}\,y_1 + x_{13}\,x_4$
60	$x_2 - 69\,x_{30}\,x_{57}$	$x_{57} + x_{15}\,y_2$
61	$x_3 - 6\,x_{52}\,x_{58}$	$x_{58} + \frac{91}{6}\,x_{15}\,y_1 + x_{20}\,y_1 - x_{15}\,x_{13}$
62	$x_4 + x_{30}\,x_{57}$	
63	$x_5 + 170\,x_{30}\,x_{59}$	$x_{59} + y_2\,(y_2 + x_{18})$
64	$x_6 + 75\,x_{30}\,x_{59}$	
65	$x_7 + 195\,x_{30}\,x_{59}$	
66	$x_8 + 2\,x_{30}\,x_{60}$	$x_{60} + x_{13}\,y_2$
67	$x_9 - x_{30}\,x_{48}$	
68	$x_{10} + 170\,x_{30}\,x_{61}$	$x_{61} + x_{17}\,y_2$
69	$x_{11} + x_{30}\,x_{61}$	
70	$x_{12} + 5\,x_{30}\,x_{62}$	$x_{62} + x_{19}\,y_2$
71	$x_{14} - x_{30}\,x_{63}$	$x_{63} + y_2\,(-2\,y_2 + x_9)$
72	$x_{16} - 195\,x_{50}\,x_{64}$	$x_{64} + \frac{1}{39}\,(34\,x_4 + 39\,x_{43})\,(-2\,y_2 + x_9)$
73	$x_{17} - x_{30}\,x_{46}$	
74	$x_{18} - x_{30}\,x_{63}$	
75	$x_{19} - 39\,x_{30}\,x_{65}$	$x_{65} + \frac{1}{13}\,y_2\,(5\,x_1 + 13\,x_4)$
76	$x_{21} - 6\,x_{30}\,x_{46}$	
77	$x_{22} + 182\,x_{30}\,x_{66}$	$x_{66} + \frac{1}{91}\,y_2\,(69\,x_1 - x_3 + 91\,x_4 - x_5)$
78	$x_{24} + x_{30}\,x_{67}$	$x_{67} + y_2\,x_6$
79	$x_{25} + 2\,x_{68}\,x_{30}$	$x_{68} - \frac{1}{2}\,y_2\,(75\,x_1 - 2\,x_2 - 2\,x_7)$
80	$x_{27} + x_{69}\,x_{70}$	$x_{69} + x_{15}\,x_{46}$
80		$x_{70} + \frac{1}{13}\,(5\,x_1 + 13\,x_4)\,(-2\,y_2 + x_9)$
81	$x_{28} - x_{46}\,x_{71}$	$x_{71} + x_{15}\,(-2\,y_2 + x_9)$
82	$x_{29} - x_{46}\,x_{70}$	

	L	R
83	$x_{31} - x_{48}\,x_{72}$	$x_{72} + \frac{1}{13}\,x_4\,(5\,x_1 + 13\,x_4)$
84	$x_{34} - x_{30}\,x_{73}$	$x_{73} + \frac{1}{13}\,x_9\,(5\,x_1 + 13\,x_4)$
85	$y_1 + x_{74}\,x_{75}$	$x_{74} + x_{30}\,y_2$
85		$x_{75} - y_2\,x_8 - 2\,x_{13}\,y_2 + x_9\,x_{13}$
86	$y_2 + 390\,x_{76}\,x_{77}$	$x_{76} + y_2\,x_{33}$
86		$x_{77} + \frac{1}{13}\,x_{50}\,(5\,x_1 + 13\,x_4)$
87	$x_2 - 75\,x_{74}\,x_{78}$	$x_{78} + x_{11}\,y_2 + 2\,x_{17}\,y_2 - x_4\,y_2 - \frac{1}{75}\,x_6\,x_{15} - x_{18}\,x_4 - x_9\,x_{17}$
88	$x_3 - 195\,x_{74}\,x_{79}$	$x_{79} - x_4\,y_2 + 2\,x_{17}\,y_2 + x_{11}\,y_2 - \frac{1}{195}\,x_2\,x_{15} - \frac{1}{195}\,x_7\,x_{15} + \frac{5}{26}\,x_1\,x_{15} - x_9\,x_{17} - x_{18}\,x_4$
89	$x_5 - x_{74}\,x_{80}$	$x_{80} - 195\,x_4\,y_2 + 10\,x_{19}\,y_2 - 75\,y_2\,x_1 + x_{12}\,y_2 - 5\,x_9\,x_{19} - 195\,x_{18}\,x_4 - 75\,x_1\,x_{18}$
90	$x_6 - 75\,x_{74}\,x_{75}$	
91	$x_7 - 195\,x_{74}\,x_{75}$	
92	$x_{10} + 195\,x_{74}\,x_{81}$	$x_{81} + \frac{5}{13}\,x_{17}\,x_1 + x_{17}\,x_4 + \frac{1}{39}\,x_4\,x_{19}$
93	$x_{11} - x_{46}\,x_{82}$	$x_{82} + x_{30}\,x_{13}$
94	$x_{12} - 195\,x_{60}\,x_{83}$	$x_{83} + \frac{34}{39}\,{y_1}^2 + x_4\,x_{30} + \frac{5}{13}\,x_1\,x_{30}$
95	$x_{16} + 195\,x_{30}\,x_{84}$	$x_{84} + \frac{1}{39}\,y_1\,(-34\,x_{32} + 39\,x_{29})$
96	$x_{22} - 2\,x_{46}\,x_{85}$	$x_{85} - \frac{1}{2}\,x_{30}\,(-2\,x_2 + 75\,x_1)$
97	$x_{34} + 4\,x_{30}\,x_{86}$	$x_{86} + \frac{1}{1560}\,y_2\,(525\,x_1 + 1560\,x_4 + 2\,x_2)$
98	$y_2 + 195\,x_{87}\,x_{88}$	$x_{87} + x_4\,x_{30}$
98		$x_{88} + \frac{1}{13}\,x_{48}\,(5\,x_1 + 13\,x_4)$
99	$x_{16} - 195\,x_{30}\,x_{89}$	$x_{89} + \frac{1}{39}\,x_9\,(34\,x_4 + 39\,x_{43})$
100	$x_{34} - x_{50}\,x_{70}$	
101	$y_2 - 195\,x_{90}\,x_{91}$	$x_{90} + x_9\,(x_{30} + x_{50})$
101		$x_{91} + \frac{34}{39}\,x_{15}\,x_4 + \frac{5}{13}\,x_1\,x_{33} + x_4\,x_{33}$
102	$x_{16} + 780\,x_{52}\,x_{92}$	$x_{92} + \frac{34}{39}\,x_4\,y_1 + \frac{5}{26}\,x_1\,x_4 + \frac{1}{2}\,{x_4}^2 + y_1\,x_{43}$
103	$y_2 - 195\,x_{30}\,x_{93}$	$x_{93} + x_{27}\,y_1$
104	$y_2 - x_{94}\,x_{95}$	$x_{94} + x_{52}\,y_1$
104		$x_{95} - x_5\,x_{15} - x_3\,x_{15} + 69\,x_1\,x_{15} - \frac{525}{2}\,x_1\,x_{33} - x_{33}\,x_2 + x_{10}\,y_2$
105	$y_2 + 170\,x_{30}\,x_{96}$	$x_{96} + x_{28}\,y_1$
106	$y_2 - 6\,x_{97}\,x_{98}$	$x_{97} + x_4\,y_1$
106		$x_{98} + 65\,x_{15}\,x_4 - \frac{589}{6}\,x_{52}\,x_{15} + x_{20}\,x_{52} - 130\,x_{33}\,x_{52} + \frac{75}{4}\,x_1\,x_{15} + \frac{1}{6}\,x_2\,x_{15}$
107	$y_2 - x_{52}\,x_{99}$	$x_{99} + 6\,x_{20}\,y_1 + x_2\,x_4 + \frac{225}{2}\,x_1\,x_4 + 390\,{x_4}^2 - 170\,x_4\,x_{52} + x_{15}\,x_{21} - 346\,x_{15}\,x_4$

3 Intermezzo: the modified BCW algorithm

Each index in the table in the previous section describes a particular step in the process of wiping out polynomials with a degree higher than three. The first 34 steps were found by hand, based on the components f, h, g and t in Pinchuk's example. The rest of the steps is found automatically by an implementation of a modification of the original BCW algorithm, which works only on monomials. In this section we present the heuristics used to find the reduction to a map of degree three after adding only 99 new variables. (Note that the previous project [4] used 106 new variables.)

Remark 3.1 We do not claim that these heuristics will always lead to a faster reduction to a map of degree three than the standard BCW algorithm. We only claim that the modified algorithm worked good with Pinchuk's example.

The main idea is that this new implementation wipes out as much monomials by adding only one or two new variables. In the notation used in the previous section, this means that we have to find suitable S_1 and S_2 such that $S = c_i S_1 S_2$. The reduction in the algorithm is implemented as a loop. We describe one step of this loop.

Algorithm 1 Main reduction loop

Require: F is an n-dimensional map and X is the list of variables in F.
  $d := \deg_X(F)$.
  if $d > 3$ **then**
    $G := F_{(d)}$ {$F_{(d)}$ is the homogeneous component of F}
    for $i = 1$ to n **do**
      if $G_i \neq 0$ **then**
        $[c, S_1] := \text{split}(G_i)$
        $S_2 := G_i/(c \cdot S_1)$
        wipe_out(G_i, c, S_1, S_2)
      end if
    end for
  else
    finished
  end if

In the main loop the important statement is $G := F_{(d)}$. We restrict ourselves to the components which have maximum degree. For each component of the original map F which has a real contribution of maximum degree, we try to find one choice of c, S_1 and S_2 in order to wipe out as much as

Algorithm 2 split(G_i)

$[c, f_1^{m_1}, \ldots, f_p^{m_p}] :=$ factor(G_i)
if $p = 1$ **then**
  one_factor(G_i, c, f_1, m_1)
else if $p = 2$ **then**
  two_factors($G_i, c, f_1, f_2, m_1, m_2$)
else
  more_factors($G_i, c, f_1, \ldots, f_p, m_1, \ldots, m_p$)
end if

Algorithm 3 one_factor(G_i, c, f_1, m_1)

if $m_1 > 2$ **then**
  $S_1 := f_1^{\lceil m_1/2 \rceil}$
else $\{G_i$ is irreducible$\}$
  throw_some_monomials_out(G_i)
end if

Algorithm 4 two_factors($G_i, c, f_1, f_2, m_1, m_2$)

if $m_1 > 1$ and $m_2 > 1$ **then**
  $S_1 := f_1^{m_1}$
else if $m_1 > 2$ and $m_2 = 1$ **then**
  $S_1 := f_1^{m_1 - 1}$
else if $m_1 = 1$ and $m_2 > 1$ **then**
  $S_1 := f_2^{m_2 - 1}$
end if
if $\deg_X(f_1) = 1$ **then**
  if $(m_2 > 3)$ or $(m_2 = 2$ and $\deg_X(f_2) > 1)$ **then**
    $S_1 := f_1 f_2$
  else $\{G_i$ has no good factors$\}$
    throw_some_monomials_out(G_i)
  end if
else if $\deg_X(f_2) = 1$ **then**
  if $(m_1 > 3)$ or $(m_1 = 2$ and $\deg_X(f_1) > 1)$ **then**
    $S_1 := f_1 f_2$
  else $\{G_i$ has no good factors$\}$
    throw_some_monomials_out(G_i)
  end if
else
  $S_1 := f_1^{m_1}$
end if

Algorithm 5 more_factors($G_i, c, f_1, \ldots, f_p, m_1, \ldots, m_p$)

```
if there exists i such that m_i > 1 then
  S_1 := f_i^{m_i}
else {all factors appear with multiplicity 1}
  let Y be a list of all the factors f_i which are single variables
  if #Y = 0 then
    S_1 := f_1 f_2 {just take the first two factors}
  else if #Y = 1 then
    S_1 := Y_1
    find f_i ≠ S_1 such that deg_X(G_i/(c · S_1 · f_i)) > 1
    S_1 := S_1 f_i
  else
    sort Y with respect to the pure lexicographical ordering {this makes
    sure that if possible, the same two variables are put together}
    S_1 := Y_1 Y_2
  end if
end if
```

possible by adding at most two new variables. Note that the algorithm does not try to reduce the complete polynomial of maximum degree in a single component before going to the next component. This is because in our actual implementation it is a pretty heavy job to filter out the components of maximum degree. Therefore we try to minimize the times this action must take place. A consequence of this choice is that in each go of this main loop, each component with maximum degree is reduced by at least one monomial.

Algorithm 6 throw_some_monomials_out(G_i)

```
let Y become the list of all variables appearing in G_i
find Y_i such that \sum_{j=1}^{p} m_j is maximal and p is minimal
    {where [c, f_1^{m_1}, ..., f_p^{m_p}] := factor(G_i|_{Y_i=0})}
split(G_i|_{Y_i=0})
```

Within the split algorithm, we show on which grounds we find a good choice for c and S_1. And because the actual wiping out is exactly as in the original BCW algorithm, the wipe_out procedure is not shown. Besides, section 2 already gives a clear description of how this is done. Obviously in the actual implementation we make sure that we never add the same factor twice.

4 Reduction to a cubic homogeneous map

Computing

$$L_{107} \circ \cdots \circ L_1 \circ F^{[99]} \circ R_1 \circ \cdots \circ R_{107}$$

gives a map $G : \mathbb{C}^{101} \to \mathbb{C}^{101}$ where $\deg(G_i) \leq 3$. We can decompose G into its homogeneous components: $G = G_{(0)} + G_{(1)} + G_{(2)} + G_{(3)}$. Since

$$G = (X + G_{(0)}) \circ (G_{(1)} + G_{(2)} + G_{(3)})$$

(where $X = (y_1, y_2, x_1, \ldots, x_{99})$) and the constant part does not influence the injectivity of the map and the constant part also does not interfere with G having a nowhere vanishing determinant of its Jacobian, we can omit the $G_{(0)}$ part without loss of generality. So we redefine $G := G_{(1)} + G_{(2)} + G_{(3)}$. By applying $G_{(1)}^{-1}$ to G we get the map

$$G' := X + G'_{(2)} + G'_{(3)}$$

where $G'_{(2)} = G_{(1)}^{-1}(G_{(2)})$ and $G'_{(3)} = G_{(1)}^{-1}(G_{(3)})$. On this G' we apply the standard BCW algorithm to get a cubic homogeneous map:

$$G'' := \begin{pmatrix} X + T^2Y + TG'_{(2)} \\ Y - G'_{(3)} \\ T \end{pmatrix}$$

where $Y = (x_{100}, \ldots, x_{200})$ and $T = x_{201}$. Since G is a $\mathbb{C}^{101} \to \mathbb{C}^{101}$-map, we get that G'' is a cubic homogeneous map $\mathbb{C}^{203} \to \mathbb{C}^{203}$.

Because the distinction between the original variables (y_1, y_2) on one side and the added variables $(x_1, \ldots, x_{201})$ on the other side is no longer important, we rename the variables to z: $y_1 := z_1$, $y_2 := z_2$ and for $i \in \{1, \ldots, 201\}$, $x_i := z_{i+2}$. The resulting vector $(z_1, \ldots, z_{203})$ will be abbreviated to Z.

It turns out that 55 components of this map G'' are of the form z_i for some i. So by conjugating with a suitable permutation P of order 2, we can arrange that the last 55 components of $G''' := P \circ G'' \circ P$ are $z_{149}, \ldots, z_{203}$. In fact the permutation $P := P_{118,156} \circ P_{133,154} \circ P_{136,152} \circ P_{146,150}$ does the trick. Note that conjugation with this permutation does not alter the fact that the determinant of the Jacobian is nowhere vanishing.

Unfortunately this map is too large to print in this paper. Especially regarded this is only an intermediate result. The reader will have to be satisfied with the construction presented above.

5 Transformation to Drużkowski maps

At this point we are halfway to our final destination. Let $F := G'''$. Now our F is of a form that it can be used as input to the GZ algorithm to find a Drużkowski map which is paired through some matrices B and C to the cubic homogeneous map G.

From a certain point of view, this is the easiest half of the trail. The algorithm is completely automated and no user-input is needed whence the computation has started. Unfortunately, the size of the encountered matrices B and D is too big for Maple to handle in a proper way. Therefore we had to apply the GZ algorithm by hand, judging carefully at each step which format to use.

5.1 Step 1

The first step in the algorithm is writing the monomials as a sum of cubic powers of linear forms of Z. During this process we build two matrices: D_0 which consists of the linear forms and B_0 which consists of the coefficients of the linear forms. By using the same order for B_0 and D_0 one gets the identity:

$$F = Z - B_0(D_0 Z)^{*3}.$$

The dimension of B_0 is 203×1941. Obviously this is also the dimension of the transpose of D_0. It took about 95 seconds to compute B_0 and about 550 seconds to compute D_0.

This figure of 1941 means that we will need a paired map $F : \mathbb{C}^N \to \mathbb{C}^N$ where $N \geq 1941$. Since B_0 need not be of full rank it is possible that we have to add some columns to B_0 in order to obtain this full rank. Of course one wants to extend B_0 at this stage 'as cheap as possible' since every column added means that N increases by 1. In the 1997 project we used the command `rank` to see whether a certain unit vector is already in the span of the columns or not. However adding one unitvector and checking whether the rank has increased or not, took approximately two days of computation time. Hence in 1997 we used a method that definitely works, but is not cheap. We just added a 217×217 identity matrix.

This time we used a different approach. We used the command `linsolve`. Basically we checked whether $B_0 X = e_i$ had a solution or not. If it did have a solution, e_i was already in the span of the columns. If it didn't, adding e_i would increase the rank of B_0 by 1. To our surprise it took Maple only 4 minutes to find out whether e_i should be added or not. And quite funny is that the `linsolve` gives the rank of B_0 as a side effect. Hence, the two day computation of `rank(`B_0`)` could be reduced to a 4 minute computation using `linsolve`! It turns out to be 145, hence we must add 58 vectors.

Of course using `linsolve` implies that each time we find a good new vector, we must augment B_0 by this vector to a new B_0'. However, augmenting a vector to a matrix of this size takes a lot of time in Maple. And we must do this 59 times. Fortunately we found another trick to avoid this problem in Maple. We simply added a 203×58 zero-matrix to B_0. This takes approximately the same amount of time as adding one vector. However, each time we find a new e_i, we only have to set one coefficient in the matrix from 0 to 1. And this takes no time at all!

The total time needed for this task was about 14 hours. To this time information we must add that we didn't use the fact that if we have to add q vectors and we have already checked $203 - q$ vectors, we know that each of the q remaining vectors must be added. If we did use this property, we would have saved about 4 hours.

In the same way we wanted to extend D_0 to a matrix of full rank. However it turns out that D_0 was already of rank 203. Hence we don't have to add anything to D_0.

At this point we have found the 203×1999 matrix B and the 1999×203 matrix D, such that

$$G = Z - B(DZ)^{*3}.$$

In particular we have found that $N = 1999$.

5.2 Step 2

Next step in the algorithm is to compute a right-inverse C of B. So we are looking for a 1999×203 matrix C with $BC = I_{203}$. Maple's `linsolve(B,I)` on the matrix level crashed due to a memory problem. However by using `linsolve` 203 times on the vector level, Maple was able to find C. It took almost 16 hours of computation time.

5.3 Step 3

The third step is computing the matrix M, the kernel of B. In the project of 1997 this was done using the straightforward command `kernel(B)`. It was computed without any problems, but ik took more than 63 hours. In order to avoid this long time, we used a different approach this time. Simply put $X := (x_1, \ldots, x_{1999})$ and compute BX. This gives a set of 203 polynomial equations in $x_1, \ldots, x_{1999}$. Solving this set gives a generator for the kernel of B. The computation time for this generator was about 20 minutes. However, this generator contains 1796 free variables. By choosing one of those free variables equal to 1 and the other 1795 equal to 0, we get a single column of M. Obviously, we must do this 1796 times. Only then we have found M, a 1999×1796 matrix. The computation time of this part of expanding the generator into the matrix took over 5 hours. Hence the method wasn't as promising after the first 20 minutes, but still a lot better than the original method leading to 63 hours.

5.4 Step 4

The fourth step consists of computing $(C|M)^{-1}$. This was the step that caused the problems in 1997; we couldn't calculate this inverse. Therefore we had to try it in a different manner this time. The key to the solution was the difference between the next two Maple statements:

- `CM:=matrix([ row1,...,row1999 ])`
- `CM:=[ row1,...,row1999 ]`

where `row1` is implemented as a list. The first option would seem to be the normal choice: we are interested in the inverse of a matrix. However the implementation of a list of lists turned out to be much more efficient. A simple declaration of a 1999 null matrix already takes hours, whereas the declaration of a list of 1999 lists of 1999 zeroes each takes only minutes. Therefore we implemented the basic Gaußelimination on lists of lists to find the inverse. And it worked fairly well: instead of not finding the answer, we found the answer after only 7 hours!

5.5 Step 5

In the draft [4] we wrote:

> The fifth and last step 'only' consists of some matrix multiplications, $A := (D|0)(C|M)^{-1}$, so we think that if the fourth step would have been completed succesfully, also the final step wouldn't have given too much trouble.

This was a mistake! Now that we managed to compute $(C|M)^{-1}$ we actually did these multiplications.

The computation of this product was completed, but it definitely took some work. The actual computation took over 273 hours, or over 11 days. Fortunately the process could be done in parallel sessions, so we didn't have to wait 11 days. We split the computation into four parts. Hence we had to wait about 3 days to get the final matrix.

In order to store this matrix as cheap as possible, we only store the 567368 elements not equal to 0. This means that the 1999×1999 matrix is filled for less than 15%. Furthermore if we look at the values in the matrix, we see that most values appear several times. In fact there are only 4055 different values in this set of 567368 non-zero coefficients. Unfortunately, the Maple file (purely text format) which contains a list of these 4055 different values and a list with the corresponding positions in the matrix is above 7Mb large. Hence it seems undoable to print this matrix in any form. (The algorithm used to collect these values used almost 10 days to build these two lists! This long computation time is a result of the fact that it is a heavy operation to add a couple of elements to a list within a long list.)

However if one is interested in this file just send a request to the author.

5.6 Sparse matrices

One of the reasons the computations presented in this section could be completed is that the matrices found are pretty sparse. The details:

matrix	# rows	# columns	total #	$\# \neq 0$	density
B	203	1999	405797	21548	5%
D	1999	203	405797	5213	1%
C	1999	203	405797	427	0%
M	1999	1796	3590204	4568	1%
$(C\|M)^{-1}$	1999	1999	3996001	63494	2%
A	1999	1999	3996001	567368	14%

6 Final remarks

Compared to the result of the 1997 project, the 1999 project gives much more satisfaction: the job was done completely. However considering the fact that the final map or its Drużkowski matrix is too large to print in this paper, also the 1999 project is not very satisfying. Unfortunately this is probably inevitable with this particular problem.

Of course the author didn't try to prove that his modification to the BCW algorithm is in fact an optimal solution. It is definitely not unlikely that someone who really investigates this particular map, comes up with a reduction that needs less than 99 new variables. Furthermore the GZ algorithm is used because it is a clear algorithm to transform cubic homogeneous maps into cubic linear maps such that injectivity properties are invariants of the algorithm. Especially the first part uses a lot of new variables: from dimension 203 we go to 1941 in one step. Perhaps this step could be investigated thoroughly with this particular example in mind.

Until then we will have to do with this example in dimension 1999!

References

[1] H. Bass, E. Connell, and D. Wright, *The Jacobian Conjecture: Reduction of Degree and Formal Expansion of the Inverse*, Bulletin of the American Mathematical Society **7** (1982), 287–330.

[2] A. Białynicki-Birula and M. Rosenlicht, *Injective morphisms of real algebraic varieties*, Proceedings of the American Mathematical Society **13** (1962), 200–203.

[3] G. Gorni and G. Zampieri, *On cubic-linear polynomial mappings*, Indagationes Mathematicae, New Series **8** (1997), no. 4, 471–492.

[4] E.-M.G.M. Hubbers, *Pinchuk's example*, preprint, 1997.

[5] S. Pinchuk, *A counterexample to the real Jacobian Conjecture*, Math. Zeitschrift **217** (1994), 1–4.