Knowledge representation and acquisition by concept lattices

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Abstract

Concept lattices are practical representations of relations. Such a lattice is based on the category theoretical notion of a Galois connection. Due to their clear mathematical background, properties of concept lattices can be proved in a neat way, in fact, calculationally.

We show how concept lattices can be used for representing knowledge, and how the recursive application of a Galois connection leads to a representation that may be useful for automatic acquisition of knowledge.

Keywords Galois connection, knowledge representation, knowledge acquisition, formal concepts, lattice theory.

1 Introduction

In this paper we consider conceptual knowledge and its representation by formal concept lattices. This lattice theoretical model was introduced about a decade ago, by Rudolf Wille ([Wil82]), for modelling the philosophical notion of a concept.

If we abstract from the details of his model, we get that a concept lattice can efficiently represent a pair of interrelated sets and, in general, any relation. In the original model of Wille, these sets correspond to the object (G) resp. attribute (M) sets which, together with a relation (R) between these sets, define a triple (G, M, R), called a *context*.

A *concept* is a pair (A, B), with $\mathsf{A} \subseteq G$ and $\mathsf{B} \subseteq M$ such that A contains all objects that share all attributes of B, and B covers all attributes common for all objects of A. So, the members of a concept mutually determine each other.

Informally, such a "construction" of a concept is called a Galois connection, in fact a particular one, called the *polar* ([Bir67]).

The set of concepts (of a context) partially ordered by "$\sqsubseteq$" is a lattice, called the *concept lattice*. For concepts $(\mathsf{A}_1, \mathsf{B}_1)$ and $(\mathsf{A}_2, \mathsf{B}_2)$ the ordering is defined by: $(\mathsf{A}_1, \mathsf{B}_1) \sqsubseteq (\mathsf{A}_2, \mathsf{B}_2) \equiv \mathsf{A}_1 \subseteq \mathsf{A}_2$ (or equivalently, $\mathsf{B}_2 \subseteq \mathsf{B}_1$).

An obvious use of such a representation, e.g. in databases, allows a subset of G (resp. M) to be given for which a corresponding subset of M (resp. G) is computed (if such exists) by searching the lattice for the "smallest" concept containing the given subset. So, for e.g. given input $\mathsf{A}' \subseteq G$ the answer is a concept (A, B) of the concept lattice, such that, $\mathsf{A}' \subseteq \mathsf{A}$ and A is "smallest". Then B is the corresponding subset of M.

A generalization of the above construction is given as follows. We have mentioned that a concept lattice is constructed from a relation. Interestingly, the result of this construction, the concept lattice, is itself a relation, too. If we take the carrier set of the concept lattice as both the object and the attribute sets, and the ordering between the lattice elements as a relation, then we have all ingredients of a new context, so the construction based on a Galois connection can be applied again.

Another possibility for generalization, which turns to be more practical ([SF94b]), is to consider the *subsets* of a concept lattice as objects and attributes, and their relation, as a basis for applying the concept lattice construction again. Clearly, in both cases, the definition of a new (abstract) context and the construction of a new concept lattice can be done time and again. That is, what we call the recursive application of a Galois connection.

A use of such a representation allows a concept sublattice to be given for which a corresponding sublattice of the concept lattice is computed (if such exists) by searching the lattice for the "smallest" sublattice containing the given one. We do not go into the issue of how "smallest" in this case can be specified. This problem has been addressed in [SF94a]. Instead, we assume that only complete matches are considered. However, we extend our model by allowing any number of concept sublattices to be given and searched for in the concept lattice.

We should also mention that some of the recursively generated lattice elements are not meaningful in the sense that one can not always find an interpretation (i.e. meaning, or explanation) for them within the original context. This observation is important because the generated lattices can be very large, due to the powerset construction (clearly, this follows from the fact that we may have to consider all subsets of a set). This indicates that certain applications of concept lattices may have to use external knowledge.

The paper is organized as follows. Section 2 recalls basic definitions and theorems of lattice theory. The terminology and much of the presentation is borrowed, with appropriate adjustments, from [ABH$^+$92]. Section 3 recapitulates basic notions and a fundamental theorem on concept lattices. Section 4 describes an extension and applications of formal concepts. Equational proofs are presented in the style introduced by W.H.J. Feijen in [DF88].

2 Basic definitions

In this section we recall some useful definitions and theorems about Galois connections. In what follows we take A and B to be sets, A and B to be set variables, and $\mathcal{A} = (A, \sqsubseteq_A)$ and $\mathcal{B} = (B, \sqsubseteq_B)$ to be posets. When this will not lead to confusion we drop the subscripts from the orderings. We let $F_1 \in A \leftarrow B$ and $F_2 \in B \leftarrow A$ to be functions. Function composition is denoted by $\bullet$.

We denote by $P.x$ the fact that the predicate P might depend on x. For predicates P and Q, that might depend on x, universal quantification, "for all x such that $P(x)$ holds, $Q(x)$ holds", is written $\forall(x : P.x : Q.x)$. The same formula without quantification and for Q a function denotes set abstraction.

We allow functions to be lifted from elements to sets. If $F : B \leftarrow A$, then for $\mathsf{A} \subseteq A$ the lifted function F is defined by $F.\mathsf{A} = (a : a \in \mathsf{A} : F.a)$.

Definition 2.1 (Galois connection) A pair of functions (F_1, F_2) is called a Galois connection iff $\forall(x, y : \ x \in B \ \wedge \ y \in A : \ F_1.x \sqsubseteq y \ \equiv \ x \sqsubseteq F_2.y)$.

Theorem 2.2 If (F_1, F_2) is a Galois connection then both F_1 and F_2 are monotonic.

Theorem 2.3 If (F_1, F_2) is a Galois connection then $F_1 = F_1 \bullet F_2 \bullet F_1$ and $F_2 = F_2 \bullet F_1 \bullet F_2$.

Corollary 2.4 If (F_1, F_2) is a Galois connection then $F_1 \bullet F_2$ and $F_2 \bullet F_1$ are idempotent.

Proof

$$
\begin{array}{ll}
 & F_1 \bullet F_2 = ((F_1 \bullet F_2) \bullet (F_1 \bullet F_2)) \\
\Leftarrow & \qquad \{ \text{ theorem 2.3, associativity of } \bullet \} \\
 & F_1 \bullet F_2 = F_1 \bullet F_2
\end{array}
$$

Theorem 2.5 If $\mathcal{A}$ or $\mathcal{B}$ are complete lattices then $F_1.\mathcal{B}$ and $F_2.\mathcal{A}$ are isomorphic complete lattices.

Later we will apply theorem 2.5 to the particular case $\mathcal{A} = (\mathcal{P}.A, \subseteq)$ and $\mathcal{B} = (\mathcal{P}.B, \supseteq)$ where A and B denote a set of objects and a set of attributes, respectively, and $\mathcal{P}$ the set-valued function powerset. This function is well-defined because the sets we will consider are all finite.

We mentioned that polars ([Bir67]) play a central rôle in concept lattices. We will define polars stepwise, starting from a set of more simple functions.

Definition 2.6 For relation $R \subseteq A \times B$ we define a function to $\mathcal{P}.B$ from A by defining for every $a \in A$: $a.R = \{b \mid aRb\}$.

Definition 2.7 For relation $R \subseteq A \times B$ we define a function to $\mathcal{P}.A$ from B by defining for every $b \in B$: $R.b = \{a \mid aRb\}$.

The functions $a.R$ and $R.b$ can be lifted to functions to $\mathcal{P}.B$ from $\mathcal{P}.A$, and to $\mathcal{P}.A$ from $\mathcal{P}.B$, respectively. These functions are called *polars* in [Bir67].

Definition 2.8 For every $\mathsf{A} \in \mathcal{P}.A$ we define the right polar as $\{\!\!\{\mathsf{A}\}\!\!\}R = \cap.(a : a \in \mathsf{A} : a.R)$.

Definition 2.9 For every $\mathsf{B} \in \mathcal{P}.B$ we define the left polar as $R\{\!\!\{\mathsf{B}\}\!\!\} = \cap.(b : b \in \mathsf{B} : R.b)$.

In order to make the formulae more readable, we will sometimes denote the right resp. the left polar of a set A, by $R^r.\mathsf{A}$ resp. $R^l.\mathsf{A}$. If $\mathsf{A} = \{\mathsf{a}\}$ is a singleton set, we simply write $R^r.\mathsf{a}$ and $R^l.\mathsf{a}$.

Theorem 2.10 The pair of polars (R^r, R^l) is a Galois connection.

3 Concept lattices

Formally, a *context* is a triple (G, M, R), where $R \subseteq G \times M$ is a relation, i.e. for $g \in G$, $m \in M$, $(g, m) \in R$ iff object g has the attribute m. In our definition of a context the sets G and M are finite. A generalization of R to an n-ary relation is found in [Wil82].

Definition 3.1 For a context the following mappings are defined in [Wil82]:

a $\mathsf{A}' = \{m \in M \mid gRm \text{ for all } g \in \mathsf{A}\}$ for $\mathsf{A} \subseteq G$,
b $\mathsf{B}' = \{g \in G \mid gRm \text{ for all } m \in \mathsf{B}\}$ for $\mathsf{B} \subseteq M$.

It is not difficult to recognize that these functions are just the functions from definitions 2.8 and 2.9 in disguise, that is, $\mathsf{A}' = \{\!\!\{\mathsf{A}\}\!\!\}R$ and $\mathsf{B}' = R\{\!\!\{\mathsf{B}\}\!\!\}$. The importance of this fact is underlined by the following theorem.

Theorem 3.2 For $R \subseteq G \times M$, $\mathsf{G} \in \mathcal{P}.G$ and $\mathsf{M} \in \mathcal{P}.M$ the functions $\{\!\!\{\_\}\!\!\}R$ and $R\{\!\!\{\_\}\!\!\}$ are a Galois connection by the following equivalence:
$\{\!\!\{\mathsf{G}\}\!\!\}R \supseteq \mathsf{M} \;\equiv\; \mathsf{G} \subseteq R\{\!\!\{\mathsf{M}\}\!\!\}$.

Definition 3.3 (Concept) A concept is a pair (G, M) with $\mathsf{G} = R\{\!\!\{\mathsf{M}\}\!\!\} = R^l.\mathsf{M}$ and $\mathsf{M} = \{\!\!\{\mathsf{G}\}\!\!\}R = R^r.\mathsf{G}$.

Definition 3.4 (Concept lattice) The concept lattice (Begriffsverband) of a concept (G, M, R), denoted as $\underline{\mathcal{B}}(G, M, R)$, is the set of concepts with the ordering: $(\mathsf{G}_1, \mathsf{M}_1) \sqsubseteq (\mathsf{G}_2, \mathsf{M}_2) \equiv \mathsf{G}_1 \subseteq \mathsf{G}_2$.

Theorem 3.5 $\mathsf{G}_1 \subseteq \mathsf{G}_2 \equiv \mathsf{M}_1 \supseteq \mathsf{M}_2$.

Proof

$$\begin{array}{ll} & \mathsf{G}_1 \subseteq \mathsf{G}_2 \\ \equiv & \qquad \{ \text{ concept } \} \\ & \mathsf{G}_1 \subseteq R\{\!\!|\, \mathsf{M}_2 |\!\!\} \\ \equiv & \qquad \{ \text{ theorem 3.2 } \} \\ & \{\!\!|\, \mathsf{G}_1 |\!\!\} R \supseteq \mathsf{M}_2 \\ \equiv & \qquad \{ \text{ concept } \} \\ & \mathsf{M}_1 \supseteq \mathsf{M}_2 \ . \end{array}$$

We recall a fundamental theorem of concept lattices, as it has been stated in [Wil82] (in fact, the first part only). A proof can be found in ([Wil82, DP90]).

Theorem 3.6 Let (G, M, R) be a context. Then $\underline{\mathcal{B}}(G, M, R)$ is a complete lattice in which infimum and supremum can be described as follows:

$$\begin{array}{ll} \mathbf{a} & \bigsqcup_{j \in J} (A_j, B_j) = ((R^l . R^r . \bigcup_{j \in J} A_j), \bigcap_{j \in J} B_j) \ , \\ \mathbf{b} & \bigsqcap_{j \in J} (A_j, B_j) = (\bigcap_{j \in J} A_j, \ R^r . R^l . (\bigcup_{j \in J} B_j)) \ . \end{array}$$

Example 3.7 (Angle, triangle, quadrangle) .

We demonstrate the construction of a concept lattice by two examples taken from plain geometry.

Our first example is the angle. In our specification we choose as objects the points P_0, P_1 and P_2, and as attributes the alignment on straight lines e and f. Alignment of a point on a line x is denoted by on_x. The angle and the Hasse diagram of its concept lattice are depicted in figure 1.

The members of the concept lattice are:

$$\begin{array}{llll} C_0 = & (\{P_0\}, \{on_e, on_f\}) \ , & C_1 = & (\{P_0, P_1, P_2\}, \{\ \}) \ , \\ C_2 = & (\{P_0, P_1\}, \{on_e\}) \ , & C_3 = & (\{P_0, P_2\}, \{on_f\}) \ . \end{array}$$

The context of the triangle is described by the three separate points, P_0, P_1 and P_2, and straight lines, e, f and g. The triangle and its concept lattice are shown in figure 2. The lattice has members:

$$\begin{array}{llll} C_0 = & (\{\ \}, \{on_e, on_f, on_g\}) \ , & C_1 = & (\{P_0, P_1, P_2\}, \{\ \}) \ , \\ C_2 = & (\{P_0\}, \{on_e, on_f\}) \ , & C_3 = & (\{P_1\}, \{on_f, on_g\}) \ , \\ C_4 = & (\{P_2\}, \{on_e, on_f\}) \ , & C_5 = & (\{P_0, P_1\}, \{on_f\}) \ , \\ C_6 = & (\{P_0, P_2\}, \{on_e\}) \ , & C_7 = & (\{P_1, P_2\}, \{on_g\}) \ . \end{array}$$

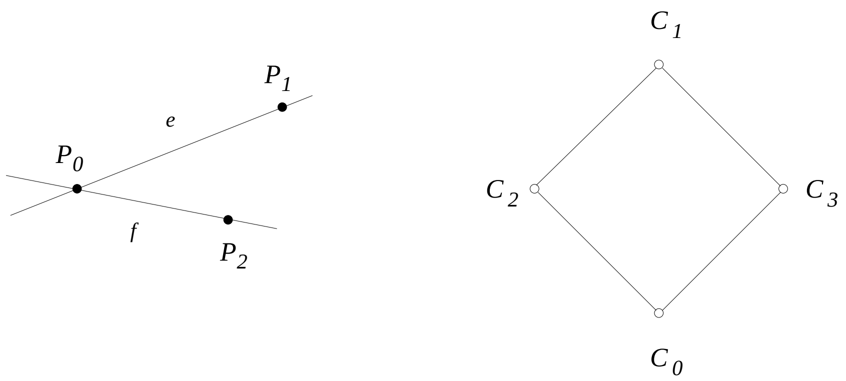

Figure 1: Diagram and concept lattice of an angle.

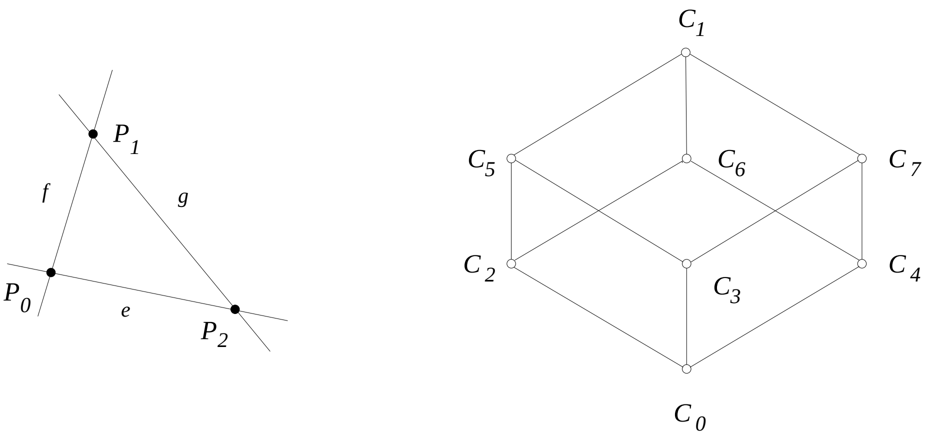

Figure 2: Diagram and concept lattice of the triangle.

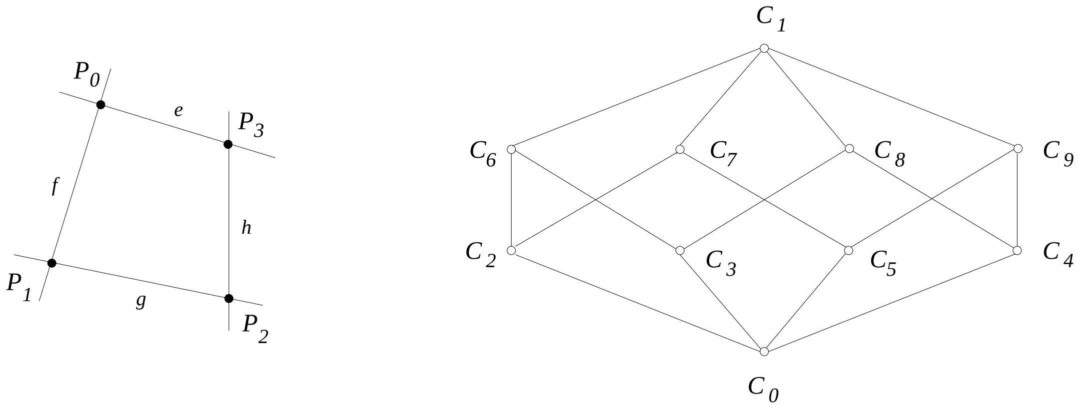

Figure 3: Diagram and concept lattice of a quadrangle.

The last example which we will need later, is the quadrangle. Following the same procedure as above, we get that now the objects are the points P_0–P_3, the attributes the alignment on straight lines e, f, g and h. The diagram of the quadrangle and its concept lattice are given in figure 3. The concept lattice has the following elements:

$$\begin{array}{rlrl} C_0 = & (\{\ \}, \{on_e, on_f, on_g, on_h\})\ , & C_1 = & (\{P_0, P_1, P_2, P_3\}, \{\ \})\ , \\ C_2 = & (\{P_0\}, \{on_e, on_f\})\ , & C_3 = & (\{P_1\}, \{on_f, on_g\})\ , \\ C_4 = & (\{P_2\}, \{on_g, on_h\})\ , & C_5 = & (\{P_3\}, \{on_e, on_h\})\ , \\ C_6 = & (\{P_0, P_1\}, \{on_f\})\ , & C_7 = & (\{P_0, P_3\}, \{on_e\})\ , \\ C_8 = & (\{P_1, P_2\}, \{on_g\})\ , & C_9 = & (\{P_2, P_3\}, \{on_h\})\ . \end{array}$$

4 Concept sublattices

We mentioned in the introduction that sublattices of a concept lattice can be useful in practical applications.

Definition 4.1 A *sublattice* of a lattice L is a subset X of L, such that $a \in X$ and $b \in X$ imply $a \sqcup b \in X$ and $a \sqcap b \in X$.

An important property which we will need later is the following.

Theorem 4.2 [DP90] For a lattice L we denote the set of sublattices by $\mathcal{P}^{sl}.L$ (the set-valued function $\mathcal{P}^{sl}$ exists as we only consider finite sets).

Then $\mathcal{P}^{sl}.L \cup \{\emptyset\}$ is a complete lattice.

Definition 4.3 ([SF94a]) Let L_1 and L_2 be concept lattices. Then L_1 is a *concept sublattice* of L_2 if there exist a surjective lattice homomorphism h to L_1 from L_2, a mapping φ to context from context, and a mapping γ to concept lattice from context, such that the following diagram commutes:

$$
\begin{tikzcd}
\underline{B}_2 \arrow[d, "h"'] & (G_2, M_2, R_2) \arrow[l, "\beta"'] \arrow[d, "\varphi"] \\
\underline{B}_1 & (G_1, M_1, R_1) \arrow[l, "\beta"]
\end{tikzcd}
$$

The next theorem gives evidences that concept sublattices arise "naturally".

Theorem 4.4 Let $\underline{\mathcal{B}}_1 = \underline{\mathcal{B}}(G_1, M_1, R_1)$ and $\underline{\mathcal{B}}_2 = \underline{\mathcal{B}}(G_2, M_2, R_2)$ be concept lattices, Let $\varphi = (\varphi_G, \varphi_M)$ where φ_G, φ_M are inverse embeddings such that $\varphi_G^{-1}.G_1 \subseteq G_2$ and $\varphi_M^{-1}.M_1 \subseteq M_2$, and furthermore $(\varphi_G^{-1}, \varphi_M^{-1}).R_1 \subseteq R_2$ holds. Then a surjective homomorphism $h \in L_1 \leftarrow L_2$ exists.

Proof

$$
\begin{array}{cl}
 & (\mathsf{A}_1, \mathsf{B}_1) \in \underline{\mathcal{B}}_1 \\
\equiv & \qquad \{ \text{ duality, concept } \} \\
 & \{\!|\mathsf{A}_1|\!\} R_1 = \mathsf{B}_1 \\
\equiv & \qquad \{ \text{ polar } \} \\
 & \cap.(x : x \in \mathsf{A}_1 : x.R_1) = \mathsf{B}_1 \\
\Leftarrow & \qquad \{ \ R_2 \supseteq (\varphi_G^{-1}, \varphi_M^{-1}).R_1,\ \mathsf{B} \supseteq \varphi_M^{-1}.\mathsf{B}_1 \ \} \\
 & \cap.(x : x \in \mathsf{A}_1 : (\varphi_G^{-1}.x).R_2) = \mathsf{B} \\
\Leftarrow & \qquad \{ \ \mathsf{A}_2 \supseteq \varphi_G^{-1}.\mathsf{A}_1,\ \mathsf{B} \supseteq \mathsf{B}_2 \ \} \\
 & \cap.(x : x \in \mathsf{A}_2 : x.R_2) = \mathsf{B}_2 \\
\equiv & \qquad \{ \text{ concept, duality } \} \\
 & (\mathsf{A}_2, \mathsf{B}_2) \in \underline{\mathcal{B}}_2 \ .
\end{array}
$$

By this we have that a homomorphism to L_1 from L_2 exists . Starting from this proof one can easily define h by a congruence θ on $\underline{\mathcal{B}}_2$ and an isomorphism between $\underline{\mathcal{B}}_2/\theta$ and $\underline{\mathcal{B}}_1$. Technical details of such a definition are omitted.

Surjectivity of h can be shown as follows. We know, by definition 3.4, that any $(\mathsf{A}_1, \mathsf{B}_1) \in \underline{\mathcal{B}}_1$ can be written as $(\mathsf{A}_1, R^l.R^r.\mathsf{A}_1)$. By assumption $(\varphi_G^{-1}, \varphi_M^{-1}).R_1 \subseteq R_2$, therefore the closure of $\varphi_G^{-1}.\mathsf{A}_1$ is the pair $(R^l.R^r.\varphi_G^{-1}.\mathsf{A}_1, R^r.\varphi_G^{-1}.\mathsf{A}_1) \in \underline{\mathcal{B}}_2$ which is a "parent" of $(\mathsf{A}_1, \mathsf{B}_1)$.

We give a "point-level" proof of this theorem, as well. The reason for considering such a proof is that the equational proof, though short and elegant, may not be satisfactory for an *implementation* of concept sublattices.

In the proof below we assume (besides those stated in the premises) that the context of $\underline{\mathcal{B}}_2$ is yielded by "minimally" extending the context of $\underline{\mathcal{B}}_1$, by adding a single object-attribute pair to it, that is, $R_2 = R_1 \cup (\{\mathsf{o}\}, \{\mathsf{a}\})$. We show how $\underline{\mathcal{B}}_2$ is constructed from $\underline{\mathcal{B}}_1$, and prove at the same time that $\underline{\mathcal{B}}_1$ is a concept sublattice of $\underline{\mathcal{B}}_2$. Initially, we let $\underline{\mathcal{B}}_2$ be the empty set.

Case 1 $\mathsf{o} \notin G_1$, $\mathsf{a} \notin M_1$.
$\underline{\mathcal{B}}_1$ and the pair $(\{\mathsf{o}\}, \{\mathsf{a}\})$, which is a new concept, are added to $\underline{\mathcal{B}}_2$.

Case 2a $\mathsf{o} \notin G_1$, $\mathsf{a} \in M_1$.
Since a is the only attribute o has, we may add $\underline{\mathcal{B}}_1$ to $\underline{\mathcal{B}}_2$, except for the concept of the form $(\mathsf{x}, \{\mathsf{a}\})$, where $\mathsf{x} \subseteq G_1$. If such concept exists in $\underline{\mathcal{B}}_1$, then a modified concept $(\mathsf{x} \cup \{\mathsf{o}\}, \{\mathsf{a}\})$ is added to $\underline{\mathcal{B}}_2$, and otherwise, a new concept $(R^l.\mathsf{a}, \{\mathsf{a}\})$.

Case 2b $\mathsf{o} \in G_1$, $\mathsf{a} \notin M_1$.
The symmetrical pair of case 2a.

Case 3 $\mathsf{o} \in G_1$, $\mathsf{a} \in M_1$.
Again, we may add all concepts of $\underline{\mathcal{B}}_1$ to $\underline{\mathcal{B}}_2$, except those of the form (x, y), where $\mathsf{a} \in \mathsf{y}$. (The other half of the proof, when $\mathsf{o} \in \mathsf{x}$ is demanded, goes similarly.)

Subcase 3a $\mathsf{o} \in \mathsf{x}$.
The unchanged concept (x, y) is added to $\underline{\mathcal{B}}_2$.

Subcase 3b $\mathsf{o} \notin \mathsf{x}$.
i) If $\mathsf{y} \subseteq \mathsf{o}.R_1$ (cf. definition 2.6) then add the modified concept $(\mathsf{x} \cup \{\mathsf{o}\}, \mathsf{y})$ to $\underline{\mathcal{B}}_2$.
ii) Otherwise (there exists $\mathsf{a}_i \in \mathsf{y}$ s.t. $\mathsf{o} R_1 \mathsf{a}_i$ does not hold).
The unchanged concept (x, y) is added to $\underline{\mathcal{B}}_2$. Furthermore, a new concept $(\mathsf{x}', \mathsf{y}') := (R^r.R^l.(\mathsf{x} \cup \{\mathsf{o}\}),\ R^r.(\mathsf{x} \cup \{\mathsf{o}\}))$ is added to $\underline{\mathcal{B}}_2$.

We observe that in all cases above $\underline{\mathcal{B}}_2$ contains all concepts from $\underline{\mathcal{B}}_1$ either unchanged or modified (a modified concept of $\underline{\mathcal{B}}_1$ is also a concept of $\underline{\mathcal{B}}_2$ and such a (new) concept is always at least as large as the old one). All other concepts of $\underline{\mathcal{B}}_2$ are constructed in subcase 3b (ii). Therefore we conclude that, functions φ and h from definition 4.3 can be given, and eventually, $\underline{\mathcal{B}}_2$ contains $\underline{\mathcal{B}}_1$ as a concept sublattice.

Example 4.5 (Angles of a quadrangle) We can observe that there are four occurrences of the concept lattice of the angle in the one of the quadrangle, as sublattices at $A_0 = \{C_1, C_2, C_6, C_7\}$, $A_1 = \{C_1, C_3, C_6, C_8\}$, $A_2 = \{C_1, C_4, C_8, C_9\}$ and $A_3 = \{C_1, C_5, C_7, C_9\}$. The definition of φ and h is simple and left to the reader.

We should point out that the specification of the quadrangle in example 3.7 contains *no* mentioning whatsoever of its angles. Nevertheless we are able to find them, due to our representation and definition 4.3.

4.6 Knowledge representation by concept lattices

In this section we will show that concept lattices may be favorably used for representing knowledge. As we have already seen the concept lattice representation of various geometrical entities, it is most simple for us to give an example of geometrical knowledge, e.g. the theorem of Thales. We should emphasize that our example is only meant to be an illustration; a full scale, realistic example may not fit in the available space.

A context and the concept lattice for this theorem may look like the one shown in figure 4 (alignment on a circle k is denoted by c_k). The theorem states that the bisecting lines at P_2 are perpendicular. For now this is *all* we have to know about this geometrical notion. At the end of this section we will briefly go into the limitations of this approach.

The lattice has members:

$$
\begin{aligned}
C_0 = & \quad (\{P_0\}, \{on_g, c_k\}) \ , \\
C_1 = & \quad (\{P_0, P_1, P_3\}, \{on_g\}) \ , \\
C_2 = & \quad (\{P_1, P_3\}, \{on_g, on_k\}) \ , \\
C_3 = & \quad (\{P_1\}, \{on_e, on_g, on_k\}) \ , \\
C_4 = & \quad (\{P_0, P_1, P_2, P_3\}, \{\ \}) \ , \\
C_5 = & \quad (\{P_1, P_2\}, \{on_e, on_k\}) \ , \\
C_6 = & \quad (\{\ \}, \{on_e, on_f, on_g, on_k, c_k\}) \ , \\
C_7 = & \quad (\{P_2\}, \{on_e, on_f, on_k\}) \ , \\
C_8 = & \quad (\{P_1, P_2, P_3\}, \{on_k\}) \ , \\
C_9 = & \quad (\{P_3\}, \{on_f, on_g, on_k\}) \ , \\
C_{10} = & \quad (\{P_2, P_3\}, \{on_f, on_k\}) \ .
\end{aligned}
$$

Let us assume that an other context is given, as well. This context and its lattice are depicted in figure 5 and figure 6. The lattice has members:

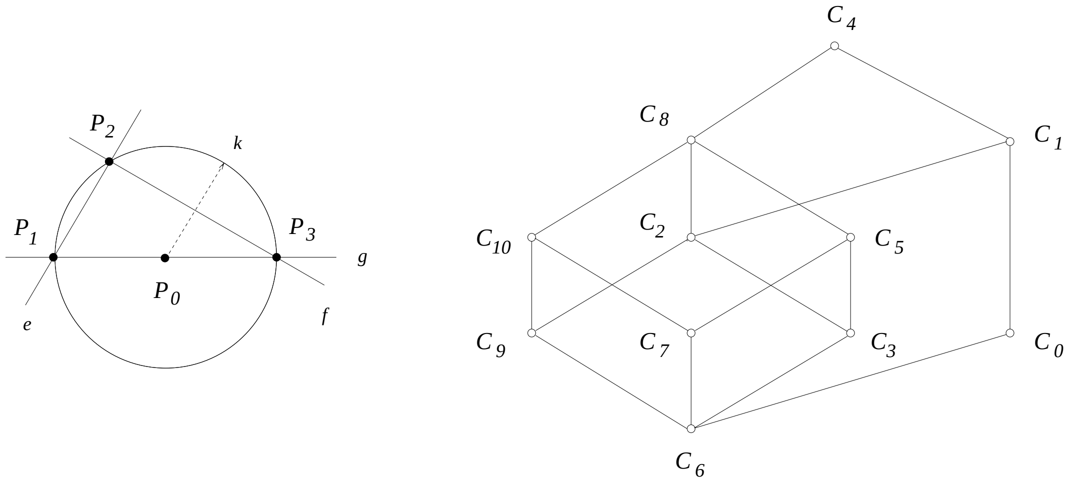

Figure 4: Diagram and concept lattice for Thales' theorem.

$$\begin{aligned}
C_0 =\ & (\{P_0\}, \{on_g, c_k\}) \\
C_1 =\ & (\{P_0, P_1, P_3\}, \{on_g\}) \\
C_2 =\ & (\{P_1, P_3\}, \{on_g, on_k, on_m\}) \\
C_3 =\ & (\{P_1\}, \{on_e, on_g, on_k, on_m\}) \\
C_4 =\ & (\{P_1, P_2\}, \{on_e, on_k\}) \\
C_5 =\ & (\{P_2\}, \{on_e, on_f, on_k\}) \\
C_6 =\ & (\{P_1, P_2, P_3\}, \{on_k\}) \\
C_7 =\ & (\{P_3\}, \{on_f, on_g, on_k, on_m\}) \\
C_8 =\ & (\{P_2, P_3\}, \{on_f, on_k\}) \\
C_9 =\ & (\{P_4\}, \{c_m\}) \\
C_{10} =\ & (\{\ \}, \{on_e, on_f, on_g, on_k, on_m, c_k, c_m\}) \\
C_{11} =\ & (\{P_1, P_2, P_3, P_4\}, \{\ \})
\end{aligned}$$

We will use our "knowledge" and prove that the angle at P_2 in figure 5 is a perpendicular. Indeed, we observe that the lattice of Thales' theorem (cf. fig. 4) is a concept sublattice of the lattice of figure 6 (in fact, only concept C_9 and its associated edges form a difference between the two lattices). It should be clear by now that our sample context might have contained any other geometrical figures, and yet we would be able to apply Thales' theorem.

The novelty of our approach is that we need not have to specify, for the given context, *how* and *when* Thales' theorem can be applied. This is "done" by the concept sublattice relation. The reader is invited to write an alternative solution to the above example in Prolog, and compare it with ours one. The differences may convince him/her about the usefulness of our approach.

The limitations of the proposed method are due to the fact that different mathematisations of a geometrical (or, in general, a philosophical) concept are

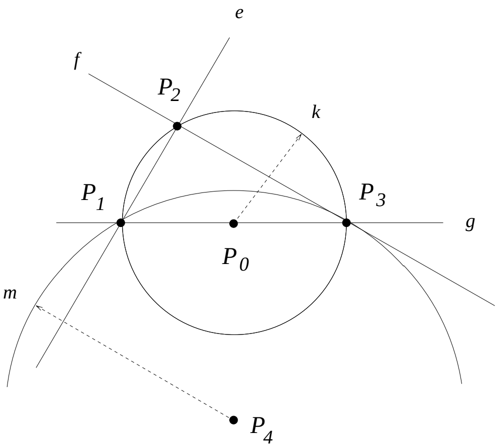

Figure 5: Sample context.

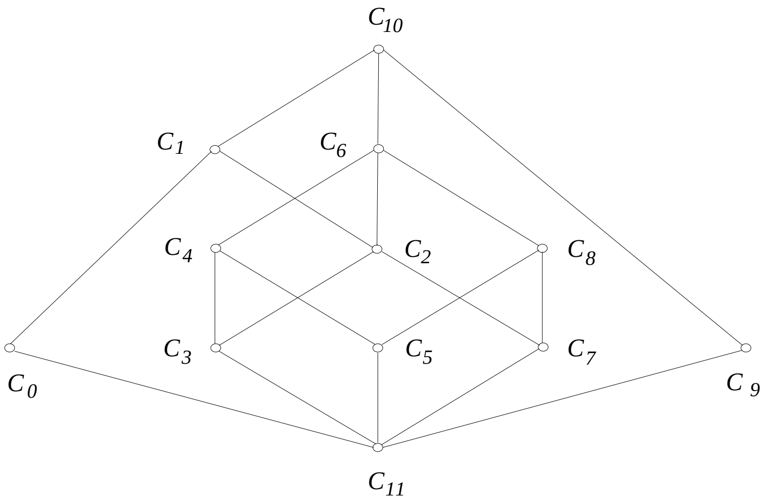

Figure 6: Concept lattice of a sample context.

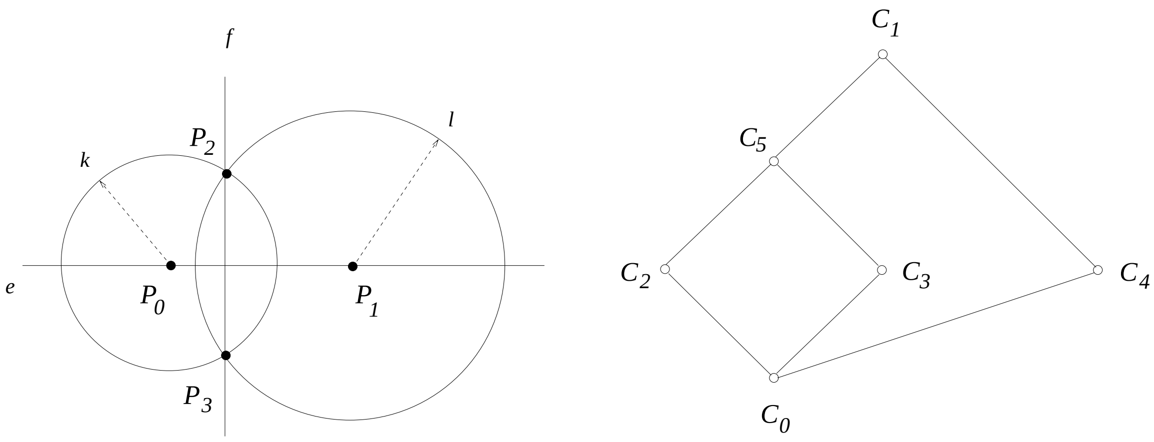

Figure 7: Diagram and concept lattice of perpendicular lines.

possible. For instance, a perpendicular can also be specified by two intersecting circles, as shown in figure 7). It is however, undecidable whether two specifications are equivalent (in general).

4.7 Knowledge acquisition by concept lattices

An obvious generalization of our lattice construction may consider a concept lattice $\underline{\mathcal{B}} = (B, \sqsubseteq)$ as a relation with, as pointed out in section 1, concepts and their related concepts as objects G_c, and attributes M_c, respectively. We have that $G_c = B$ and $M_c = B$, and for all $\mathsf{a}, \mathsf{b} \in B$ the relation R_c is defined by: $\mathsf{a} R_c \mathsf{b}$ iff $\mathsf{b} \sqsubseteq \mathsf{a}$ and there is no element x in B such that $\mathsf{b} \sqsubseteq \mathsf{x} \sqsubseteq \mathsf{a}$. As we have now all ingredients of the context (G_c, M_c, R_c) we can construct a concept lattice again.

We mentioned a more fruitful generalization which considers the sublattices of a concept lattice as objects and their related sublattices as attributes. Recall that, by theorem 4.2, the set of sublattices with the inherited ordering is a complete lattice.

Definition 4.8 For a concept lattice $\underline{\mathcal{B}}$ and sublattices $\mathsf{a} \subseteq \underline{\mathcal{B}}$ and $\mathsf{b} \subseteq \underline{\mathcal{B}}$, we say a is related to b by relation R_s, i.e. $\mathsf{a} R_s \mathsf{b}$, iff $\exists \mathsf{x} \in \mathsf{a}$ and $\exists \mathsf{y} \in \mathsf{b}$ such that $\mathsf{x} R_c \mathsf{y}$ holds.

Definition 4.8 can help us to find all different (abstract) conceptualizations of the input data. Due to space limitations, technical details of selecting a single conceptualization are omitted from the paper.

In a practical application one is interested in conceptualizations which only assume sublattices that are either given, or have a meaningful interpretation in some context. The next example illustrates such a case.

Example 4.9 (Quadrangle revisited) We have seen that the lattice of the quadrangle contains the sublattices A_0–A_3. Let us take them now as our objects and as attributes, the sublattices related to them. In order to make the formulae more readable, we will introduce the prefix *with* for a named subset when used as an attribute. Eventually we get the relation: $\{(A_0, \text{with } A_1)$, $(A_0, \text{with } A_3)$, $(A_1, \text{with } A_0)$, $(A_1, \text{with } A_2)$, $(A_2, \text{with } A_1)$, $(A_2, \text{with } A_3)$, $(A_3, \text{with } A_0)$, $(A_3, \text{with } A_2)\}$.

By applying the Galois connection from section 3 again we get the concepts:

$$\begin{array}{rl} C_0 = & (\{A_0, A_1, A_2, A_3\}, \emptyset) \ , \\ C_1 = & (\{A_0, A_2\}, \{\text{with } A_1, \text{with } A_3\}) \ , \\ C_2 = & (\{A_1, A_3\}, \{\text{with } A_0, \text{with } A_2\}) \ , \\ C_3 = & (\emptyset, \{\text{with } A_1, \ldots, \text{with } A_3\}) \ . \end{array}$$

Although this lattice is isomorphic to the one of the angle (cf. figure 1), it has now a completely different meaning: a quadrangle conceptualized as *pairs of angles* ! Indeed, we observe that C_1 and C_2 (those concepts that do not contain the empty set) has a pair of angles as object resp. attribute sets.

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