Mathematica Utens

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Abstract. We call our innate ability for computation the 'mathematica utens'. This paper is an attempt to introduce a cognitively based semiotic model for the concepts of such 'naive' mathematics. We argue that (naive) mathematics is a level of knowledge representation mediating between (naive) logic and natural language.

1 Introduction

In recent years we developed a theory of knowledge representation [2] which by virtue of its cognitive basis may bridge the gap between the 'naive' and formal concepts of knowledge. Why is such a link important? One of the reasons is the inherent potential of the cognitively based approach for efficiently modeling knowledge in domains that are closely related to perception. An example is natural language, and its model introduced in [8] which has been shown to be linearly complex [9]. But there is also another reason that could be equally important. If, indeed, knowledge arises via cognition from the observation of 'real' world phenomena, then this feature must hold for knowledge generated formally, via computation, as well. We argue that such formal 'knowledge' may become 'real' knowledge, if it is interpreted as such, that is, as a sign of some (potentially) observable phenomenon.

This paper is an attempt to prove that our framework can be successfully applied to the mathematical domain, in particular, to the concepts of 'naive' mathematics. Additionally we argue that (naive) mathematics is a level of knowledge representation mediating between (naive) logic and natural language. Finally we show that the mathematical notion of 'infinite' and the prototypical concepts of human knowledge may share a common representation.

In the definition and classification of signs we follow Peirce's semiotics [4]; with respect to a theory of cognition we refer to categorical perception [5].

1.1 Cognition and signs

Historically, Peirce's semiotic theory initially has been centered around his three types of signs: the icon, index and symbol. Later he completed his list to include nine types, and defined a hierarchy of signs. Our research has revealed that those additional types of signs play an important role in sign recognition, which is a cognitively based *process* of *re-presentations*. The essence of this process is summarized in the next section.

Sign recognition Signs arise from the observation of events, which in turn are due to a change in the 'real' world. The occurrence of a change triggers the brain which samples the physical stimuli in a percept. The comparison of the current percept with the previous one enables the brain to distinguish between two sorts of qualities: one, which was there and remained there, which we call a *state*; and another, which, though it was not there, is there now, which we call an *effect*. State and effect qualities and, ambiguously so, their sets are respectively denoted by a and b. From the logical point of view, the comparison of percepts involves the application of the dual operations: 'and', and 'inhibition' (or 'relative difference'). From the semiotic point of view, such a comparison is an interaction between qualities which are signs: a *sign interaction*.

The input qualities trigger the memory, which in turn generates a response, representing memorized state and effect qualities. Such qualities, as well as their sets, will be ambiguously denoted by a' and b'. Because the memory response can signify the input either in the sense of agreement('$*$'), or possibility('$+$'), an observation is finally represented, respectively, by the observed and complementary state ($A=a*a'$, $\neg A=a+a'$), and effect ($B=b*b'$, $\neg B=b+b'$), collectively called the *input signs*. By interpreting '$*$' and '$+$' as the logical operations 'and' and 'or', respectively, we get the logical meaning of these signs.

We argue that the ground for any re-presentation is our ability for recognizing similarity via *comparison*. An example for such an interaction is the one between previous and current percept, or between input and memory. By applying comparisons recursively, the brain brings forth a process of sign interactions generating increasingly better approximations of the final proposition of an observation. We maintain that such approximations have the aspects of Peirce's types of signs and can be interpreted, from the logical point of view, as Boolean logical functions. We will capitalize on this relation, and refer to a (type of) sign by means of its Boolean logical denotation.
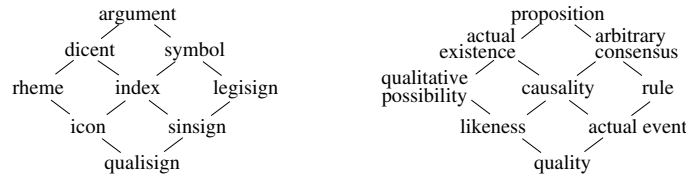
argument

dicent   symbol

rheme   index   legisign

icon   sinsign

qualisign

proposition

actual existence   arbitrary consensus

qualitative possibility   causality   rule

likeness   actual event

quality

Fig. 1. Peirce's classification of signs and aspects

The process of sign recognition proceeds as follows. First, the observed input signs are represented via *sorting*, from the point of view of similarity and simultaneity ($A+B$, $A*B$). Next, the relative difference of these signs is signified by the abstract denotations of the observation ($A*\neg B$, $\neg A*B$, and $A*\neg B+\neg A*B$). Also the complementary input signs (*context*) are represented in the sense of sorting ($\neg A+\neg B$, $\neg A*\neg B$). Subsequently, those abstract denotations are complemented by the meaning of the context ($A+\neg B$, $\neg A+B$, and $A*B+\neg A*\neg B$),

and a final proposition sign is generated via predication (A *is* B). In [8], we pointed out that each of these re-presentations amounts to a type of comparison sign interaction.

Peirce's signs and their aspects are depicted in fig. 1. The Boolean interpretation of the types of signs and the process of sign recognition, that we call the innate or '*naive*' *logic* of the brain, is displayed in fig. 2 (a horizontal line denotes a sign interaction). Complex phenomena can be recognized recursively, via nested observations [2]. The proposition sign, A *is* B, of a nested phenomenon is represented *degenerately* in the nesting phenomenon as a (complementary) qualisign, which is a subset of A, or B.

Characteristic to our model of the process of sign recognition is that the *interpreter* is the brain as a *biological system*. The signs recognized are beneath the limen of intellect, and are only preliminary with respect to the signs that appear in our consciousness. We hold that the types of such signs could be identical to those defined by Peirce in his decadic classification [11]. But further research is needed to reveal if this conjecture[1] might be true.
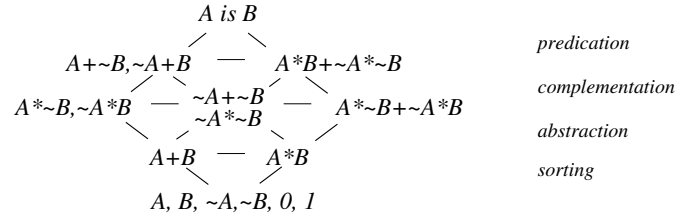

                                    A is B

    A+~B,~A+B   ——   *A*B+~A*~B*   *predication*

A~B,~A*B* —— *~A+~B* —— *A*~B+~A*B*   *complementation*
                 ~A~B*
                                                 abstraction
           A+B —— *A*B*
                                                 sorting
        A, B, ~A,~B, 0, 1

Fig. 2. The classification of Boolean signs and sign interactions

Domains and interpretations In past years we introduced various applications of our theory of knowledge representation, amongst others, a model for the semiosis of (naive) propositional logical, syllogistic and reasoning signs. Such signs, which comprise our logical faculty that we call the 'logica utens' [10], play an important role also in the present work. As due to lack of space we cannot expand on these models, we will assume some familiarity with their basic concepts.

In this paper we argue that 'naive' mathematics emerges from the logica utens due to the brain's mathematical faculty. As mathematical concepts are signs we may talk about the 'real' world of mathematics. We hold that akin to the signs of other knowledge domains also mathematical signs arise via sign interactions, reinforcing our conjecture that all human semiosis could be based on a single type of process.

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[1] inspired by Gary Richmond (pers.comm., 2003)

The model of mathematical signs introduced in this paper heavily relies on the representation of memory signs. The properties of such signs can be recapitulated as follows [3]. We assume that the state and effect qualities of earlier observations are organized by the memory in collections and chains of neurons, respectively. The memory response triggered by the actual input state and effect is represented, respectively, by an *average value* (of a collection), and a *dense domain* (of a chain) of stored qualities. By denseness we mean that after sufficiently many observations of a type of effect, the brain is capable of representing any measure of that effect by a relative value of its domain. For example, after various perceptions of different weights like 1kg, 2kg, etc., the observation of 1.5kg is represented by a sign pointing in the domain of weight-effects, between the values denoting 1kg and 2kg. If the chain representation of a domain does not contain a neuron corresponding to the input effect interpreted as a measure, then the chain is adjusted by including a yet unused new neuron (and quality), thereby keeping the domain dense.

2 The ontology of cardinality as a sign

In this section we introduce a model for the recognition of cardinality as a sign. In the first part we discuss the physiological grounds for counting, in the second we show how such abilities may contribute to the abstract conception of numbers.

2.1 Counting abilities

Recently, Nieder at al. [7] reported the discovery of number-encoding neurons in the brain. Such neurons fire maximally in response to a specific preferred number, correctly signifying a wide variety of displays in which the cues are not confounded. For instance, one such neuron might respond maximally to displays of four items, somewhat less to displays of three or five items, and none at all to displays of one or two items. The number-encoding neurons are able to recognize the number of *similar* items[2] from 1 up to 5, but the representation of number gets increasingly fuzzy for larger and larger numbers. Many neurons fire selectively a constant 120ms after display onset, whatever the number on the screen indicating that the neurons 'count' without counting (i.e. without enumerating the items serially). The evolution of number-encoding neurons may entail superiority of a species, as knowledge about the number of preys, or predators could be crucial. An exotic example for a potential evidence of a dual evolution are the stripes of the zebra. The intertwining stripes of a group of zebras can make the observation of their number troublesome.

The sign of the number-encoding neurons is *iconic* whereas the mathematical conception of cardinality is *symbolic*. The intermediate *indexical* concept, linking the iconic and symbolic representations, is the concept of ordering. In this paper we hold that the three types of number signs can be uniformly modeled.

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[2] An item is a collection of qualities. Similarity holds trivially for a singleton.

The results of [7] indicate that the brain has a counting faculty, and precisely where. But the signal of the number-encoding neurons is vague already for low numbers contradicting the common experience that the brain is able to accurately stipulate cardinality (up to a limit) without symbolic counting. Therefore, in this paper we assume that number-encoding neurons function analogously to the receptors of the senses, for example, the color receptors of the eye. The perception of a color is independent from the number of the receptors simultaneously discharged. Their signal will signify the same color, but possibly dimmer, or brighter. We hold that the perception of cardinality follows the same principle. Each of the similar items of the input possesses a cardinality quality enabling the recognition of the collection of those items, via categorical perception [5], as a number.

As cardinality arises in the brain and not in the senses, the perception of such quality requires a higher level process interpreting the input as a *number-phenomenon* and recognizing its cardinality as a number. Notice here the ambiguous use of 'cardinality' as a quality arising for each similar item of the input, and as a (cardinal) number characterizing the input as a whole.

In our model we assume that the brain by virtue of its counting ability is able to distinguish the physical input in three types. The observed state can be either:

- one
  something whole; the number-encoding neurons are not active; cardinality does not arise
- not_one
  a small multitude; some of the number-encoding neurons are active; cardinality does arise, and can be recognized as a number
- many
  a large multitude; the number-encoding neurons are all active; although cardinality does arise, it cannot be recognized as a number.

The boundary between one, and not_one is typically 1 or 2 ('chunking' is not considered in this paper). A linguistic evidence for the latter is the distinction between singular, two, and plural, for example: one, both, but all three, all four etc. Here, singular and two refer to a small multitude which is minimal. The boundary between not_one and many could be between 5 and 9, conform the size of the working memory [6]. In what follows, we will first analyze the important case of not_one. We will return to many in sect. 3.2.

2.2   The case of not_one

As cardinality as a quality arises in the brain, it can be represented as a memory sign. Because it signifies similarity, a property which is independent from, hence not part of the input effect, it can be modeled by an *a'* memory sign; and because it does not refer to the primary (i.e. observed) meaning of the input, it must be complementary.

We argue that input phenomena are always recognized, first, with respect to their primary meaning. But if the final proposition sign of the observation is found unsatisfactory, for any reason, then the brain may seek via abduction another interpretation, and try to recognize the input as a number-phenomenon, for example.

Following [10] abduction can be modeled by interpreting the *difference* between a and a' as an effect conceptually changing a into a'. Such interpretation can be said to involve a 'shift' of focus from a to its relative difference with a': we take a *different* look at the observed phenomenon. The difference between a and a' can be computed analogously to the generation of the input qualities as signs: those a and a' qualities which are related to each other in the sense of agreement define a *new* $a := a'_o * a_o = a' * a$; and those in the sense of possibility a *new* $b := a'_c \setminus a_o + a'_o \setminus a_c + a'_c \setminus a_c = a' \setminus a$ (a subscripted o and c denotes an observed and a complementary quality, respectively; "$\setminus$" stands for relative difference). The above process is completed by merging the input and abducted qualities together, type-wise.

Abduction can be modeled by a feedback of a and a', to the unit generating the input signs from the physical stimulus via comparison. Notice that a similar feedback of b and b' may not be effective, as b' being a dense domain, a meaningful difference cannot be defined. The revised model of perception (first introduced in [2]) is depicted in fig. 3. In this diagram, the contents of the current percept is written into the previous percept after some delay following input sampling. The control signal (ctrl) is used to selectively inhibit either the feedback, or the physical input qualities, respectively, in 'normal' and 'abduction' operation mode. The latter can be activated if 'normal' recognition of the input fails.
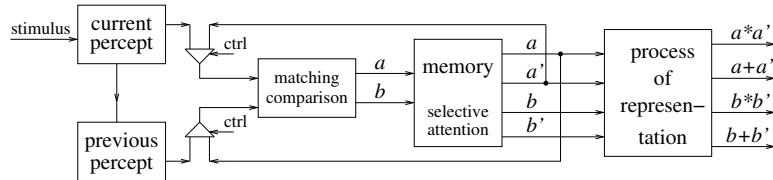

Fig. 3. The revised model of perception

2.3 Recognition of number-phenomena

We introduce a model for the recognition of number-phenomena on the basis of the *iconic* conception of cardinality. Our working example will be the observation of a multitude of similar cubic bodies. The perceived input and the memory response qualities are defined as follows (observed and complementary qualities are merged): $a=$ "cube"; $a'=$ "memory sign of cube", "memory sign of cardinality"; $b=$ "cubic form", "size"; $b'=$ "memory sign of cubic form", "memory sign

of size". The signs representing the relation between the input and memory are: $A=a*a'=$ cube; $\neg A=a+a'=$ cardinality; $B=b*b'=$ form; $\neg B=b+b'=$ size.

We assume that the proposition sign characterizing this input, e.g. 'some cubes are there', is found unsatisfactory and that via feedback new input qualities are abducted. Such *new a* is a subset of the perceived state qualities, as the meaning of a is contained in the average meaning of the (observed) a' qualities, by definition. Such *new b*, however, is an effect transforming the perception of a cube as some 'thing' to its perception as a cardinality property. The abducted new input qualities generate a memory response, but this is omitted in the paper.

The signs representing the relation between the input and the memory, following abduction, amount to the qualisigns of the observation as a number-phenomenon: $A=$ cube (a cubic body), $\neg A=$ unit value (for counting cubic bodies); $B=$ cardinality (as an appearing new property), $\neg B=$ growth (cardinality increment corresponding to the unit value, as a property). The recognition of the input as a cardinality proceeds as follows (below we will directly refer to the signs recognized, by means of their Peircean types).

The icon $(A+B)$ signifies the input qualities as constituents (similar items). The sinsign $(A*B)$ represents the co-occurrence of those qualities as an event. Because cardinality may arise for any countable input, the sinsign can be said to characterize the input as a number-event, independent from the type of similar objects signified.

The rheme $(A*\neg B, \neg A*B)$ synonymously denotes the general meaning of cubic 'things' and the abstract concept of cardinality referring to any such 'thing' (number base). The legisign $(A*\neg B+\neg A*B)$ refers to the rule-like relation between those general concepts, telling how the number of such cubic 'things' can be determined in general by means of their cardinality meaning, for example, by conceptually accumulating the cardinality qualities in a set representing a measure (rule of counting). Such procedure needs a unit of counting, synonymously denoting an increment value, and incrementation (by such a value) as a property. For example, small cubes can be counted 1/10 each, meaning that ten of such cubes make up a large one. Such value and property define a type (incr), representing complementary information, or the *context*, of a number-phenomenon as an index sign $(\neg A+\neg B, \neg A*\neg B)$.

The complementation of the abstract meaning of the rheme, by the indexical meaning of the context, yields the recognition of the input as an actual multitude $(A+\neg B)$ and cardinality $(\neg A+B)$, synonymously referring to the *subject* of the observation. Analogous complementation of the legisign obtains the *predicate* of the observation $(A*B+\neg A*\neg B)$ denoting an algorithm for the computation of B number of A entities, using type information due to $\neg A$ and $\neg B$. Finally, the interaction between the actual multitude and counting algorithm is interpreted by counting (in the iconic sense) the similar items signified by the subject. This is represented by a proposition sign characterizing the input as a number. The types of number signs are displayed in fig. 4.
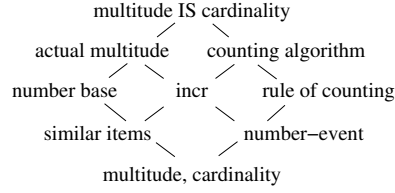
multitude IS cardinality

actual multitude   counting algorithm

number base   incr   rule of counting

similar items   number–event

multitude, cardinality

Fig. 4. The types of number signs

2.4 Inclusion ordering

We argue that the *indexical* conception of cardinality enables the brain to order different multitudes without symbolic counting. Such ability has been shown to be present in children already at the age of two [1].

A sample ordering phenomenon is the following. Assume there are two collections of cubes, on the left- and right-hand side of a separating line, and that the task is the recognition of the relation between their cardinalities. Additionally we assume that both collections have been recognized already as a cardinality, and that now we are concerned with the interpretation of the number of the cubes on the right-hand side in the *context* of the number of the cubes on left-hand side. Ordering as a sign may arise as follows.

The proposition sign of the left-hand side collection ($prop_l$) is stored in the memory. The subsequently recognized proposition sign of the right-hand side collection ($prop_r$) is represented degenerately as a value (n_r): $a_o := n_r$; $a'_o := n_r$. Then, $prop_l$ is fetched from the memory and represented analogously (n_l): $a_c := n_l$; $a'_c := n_l$. Via feedback, the previous and current percepts are replaced, respectively, by a_o, a_c, and a'_o, a'_c. New qualities are generated: $new\ a := a' * a = a'_o * a_o = n_r$; $new\ b := a' \backslash a = a'_c \backslash a_o + a'_o \backslash a_c + a'_c \backslash a_c = n_l \backslash n_r + n_r \backslash n_l$. From the mathematical point of view, the relative difference, $n_r \backslash n_l$, amounts to subtraction, which can be interpreted as an ordering sign, more specifically, as '<', only if $n_l < n_r$ (remember that by comparing the previous and current percepts, in the sense of relative difference, those qualities can be represented which were not there, but are there now). Similarly, $n_l \backslash n_r$ can be interpreted as '>', only if $n_l > n_r$. If both signs arise, then n_l and n_r must be equal ('='). The recognition of the new qualisigns eventually yields the *inclusion relation* of the right-hand side collection with respect to the left-hand side one (which is either one of less, equal, or greater), as a proposition sign.

2.5 Symbolic number signs

What makes the *indexical* concept of cardinality important is that it underlies the *symbolic* one. This can be illustrated by the indexical level rheme sign which by separately representing the input as a general state ($A * \neg B$), and an abstract cardinality effect ($\neg A * B$), creates the ground for the symbolic concept of counting any 'thing'. The primary symbolic numbers arise via learning, from cardinalities that can be precisely recognized (cf. cardinal numbers).

Let us extend our example above, by the task of rearranging the cubes such that, in the end, there are equally many cubes on both sides of the separating line. Children not possessing the symbolic concept of numbers tend to solve such a task by first collecting all cubes on one side of the line, and then, by moving the cubes one-by-one, estimating their number on both sides and determining their equality. Those familiar with symbolic numbers attain the same goal simply by counting. If rearranging is supervised, and the action of moving a cube is repeatedly followed by the articulation of "one cube", the ordering operation and the referencing to the cubes may become rule-like associated with the numeral "one". By executing the operations for many cubes and also for other objects, the meaning of "one" can be abstracted, prototypically denoting some general 'thing' (or state), which is 'counted'. In general, a finite number of such numerals can be memorized analogously.

The prototypical meaning of symbolic numbers arises by memorizing (in different observations) the state qualities which are found similar. By computing an average value for such qualities we may get prototypical concepts like the "memory sign of (a) cube" introduced in our earlier example, in sect. 2.3. Averaging, like domain formation, is a *generalization* process underlying the prototypical meanings of the memory.

Complex multitudes can be recognized sequentially, via nested observations, by focusing on contiguous parts of the input which can be individually recognized as number-phenomena. Clearly, such multitudes can be more conveniently counted by using symbolic numbers. The rules of symbolic counting are subject to learning akin to language. As such rules are also similar to those of language, a model for the semiosis of symbolic numbers can be defined easily, for example, on the basis of the language model introduced in [8].

3   The 'real' world of mathematics

We hold that the mathematical 'universe' consists of phenomena, which like 'real' world phenomena arise via interaction between a state and effect, generating a next state. What makes cardinality effects specific is their potential for interacting with *any* state. Whereas all 'real' world phenomena can be characterized (in the philosophical sense) as an 'existence', those of the mathematical universe as a 'cardinality' event. In the mathematical world, all state(a) is inherently related to a cardinality effect(b), hence via interaction the concept of 'a-number' can arise. Such number signs as states can be found *similar*, therefore countable, in a subsequent number-phenomenon, and can be re-presented as a state, recursively. Consequently, in the mathematical 'world', also effect qualities can have the meaning of a state, although degenerately only.

An illustrating example is the function $sin(\_)$, which is an effect, but which is represented as a state in the symbol phenomenon $sin^2(x)$. Such flexible use of signs is not characteristic for language, which introduces different representations for the state and the effect meanings of a denotation. For example, the verb run which is an effect, can be interpreted as a state, but only after 'converting' it to a

noun, thereby removing its verbal relational needs. Mathematical signification is free of such rigidity of re-presentation. The effect meaning of $sin(\_)$ is preserved in $sin^2(x)$ such that it still can map its parameter, which is a state (x), to its image, which is another state $(sin(x))$. We argue that the rigidity of language could be a consequence of the automatism characterizing the use of language rules, for example, in syntactic signification.

Inasmuch in the mathematical world any state and cardinality effect can interact, the generation of interpreting (new) states is unconstrained. But mathematical signs are a 'projection' of the holistic meaning of signs (mathematical signs are signs from the mathematical point of view). Therefore the mathematical world must be part of the 'real' one, in which interactions are *constrained*, meaning that not every quality can interact with another. Accordingly, in mathematics, in order to avoid the danger of a chaos, the correctness of interactions has to be verified, for example, by demanding 'well-formedness' of the state and effect qualities. To this end, mathematics introduces the meaning of types. In the next section we argue that the mathematical concept of a *type* arises via refinement, from the indexical level concept of the increment (incr).

3.1 Mathematical types

The correctness of a mathematical operation can be determined by means of contextual information about the type of the input state and effect qualities. From the semiotic point of view the operation is *well-formed* if the index sign of the input exists, in which case, $\neg A + \neg B$ signifies the compatible types of the input state and effect, and $\neg A * \neg B$ the type of the yield of the operation. The dicent and symbol sign of a mathematical phenomenon, which arise, respectively, from the recognized rheme and legisign, via complementation by the index sign, contain all information necessary for accomplishing the mathematical operation. Because $\neg A$ and $\neg B$ signify the relation between the input and memory, in the sense of possibility, also mathematical index signs represent the observed phenomenon only in a general sense, conform their meaning as a type.

It must be clear from this analysis that the index signs are the *key* to mathematical sign recognition. That, in the mathematical world, effects can have the meaning of a state, in particular, in the case of nested phenomena, brings about the need for a specific interpretation of the relation between states and effects. For example, a function can have any number of parameters that too are functions. As such nested functions do preserve their effect meaning, we have to find out the type of a nested sign interaction. Such types are the meaning of mathematical index signs: if the index sign of an input phenomenon exists, then the observed mathematical operation can be meaningful.

The basic type of cardinality, 'natural' number, follows from the default meaning of incr, which is enumeration. Other types are due to their specific number generation rules. For example, the symbolic number-phenomenon "1,5" can denote, context dependently, a pair of natural numbers, or a single rational. In sum, the meaning of a type of number is collectively defined by a number prototype (cf. general state) and a number generation rule (cf. prototypical effect).

We hold that also the formal mathematical types of number like natural ($\mathcal{N}$), rational ($\mathcal{Q}$) and real ($\mathcal{R}$), arise from the 'naive' mathematical concept of a type, via the iconic, indexical and symbolic re-presentation of the 'natural' meaning of incr, respectively, by a constant, a quotient, and a process.

3.2   The conceptualization of infinite

One of the most contradictory concepts of mathematics is the one of *infinite*. In this paper we hold that memory signs are 'natural' representations of such multitudes. Indeed, a' memory signs denote an averaged value potentially referring to *all* qualities related to the input state(a); b' memory signs refer to a domain possessing *all* measures possibly signified by the input effect(b). For example, a'=chair denotes the prototypical meaning of any seat that can be functionally sat on like an armchair or a footstool; b'=run prototypically refers to any locomotion by foot be it slow or fast.

This potential for representation by the memory allows a twofold interpretation of the relation between the input and memory qualities. According to the first, $A=a*a'$ is the meaning of a completed by a'; $B=b*b'$ is the meaning of b interpreted as a measure of b'. In both cases, a reference is made to an *individual* quality, which is a value. Following the second, $A=a*a'$ denotes a collection of qualities selected by a, which is averaged; and $B=b*b'$ a sub-domain of b' indicated by b, which is a collection of qualities organized in a chain. Now, in both cases, a reference is made to a collection containing all qualities related, but this reference does not provide access to those qualities as individual elements. In this interpretation, A and B can be said to include the 'naive' meaning of infinite.

How does infinite emerge as a sign? Earlier we mentioned that the brain, due to its number-encoding faculty, is capable of distinguishing the input state in three types: one, not_one, and many. In the last case, the input multitude possesses a cardinality quality, but no increment. We hold that such multitudes can be represented as a number, by using the 'infinite' interpretation of memory signs. Indeed, as the individual elements of the collection of qualities referred to by an average value and a domain cannot be accessed, then clearly there will be no increment (incr), implying that the multitude in question is not countable. In our model, the cardinality of such a multitude is denoted by the number sign 'infinite'.

The above relation between the concept of infinite and the ambiguous meaning of memory signs is also supported by linguistic evidence. For example, in the utterance 'the police are going to lunch', the subject is ambiguously interpreted as something whole, and as a set of individual persons. This analysis underpins our conclusion that the concept of infinite, and the prototypical notions of the memory like house, dog etc. could be related, implicating that memory signs, in general, may include the meaning of a 'kind of' infinite, potentially.

Returning to the mathematical signs, let us finally mention that also the notion of a *formal variable* could be explained on the basis of our concept of a general state. Via abstraction, a prototypical state meaning can be recognized,

denoting a collection having no reference to the contained memory qualities. Such a sign can be called a 'formal' sign: a sign, which is about the form.

3.3   The conceptualization of naught

Cardinality as a quality arises if there are similar items in the input. From this it follows that we cannot perceive naught as a quality, except via inferencing. Let us explain this by an example.

Assume the observation of a sample phenomenon having no cardinality meaning, but which we nevertheless try to recognize as a number-phenomenon. As now the observation is not due to a cardinality effect, we may abductively infer that there must have been 'something' but which has disappeared. If that 'something' may refer to a countable state, its cardinality can be denoted in the actual observation by naught or zero. In sum, zero is a hypothetical number sign, not having the meaning of cardinality as a quality, nor the meaning of an increment.

We argue that zero and infinite are dual concepts. The concept of infinite includes the meaning of a cardinality, but no increment, hence the notion of infinite is synonymous to the meaning of a multitude classified as one. But as symbolic numbers, zero(0) and one(1) are, respectively, the unit elements of "+" and "∗" which in turn are *dual* representations, in the mathematical sense, of the sign interaction *sorting*, justifying our claim.

4   The secondness of mathematics

We maintain that, if logic is a first, then mathematics a second, and language a third. Notice that by logic we mean innate logic, by mathematics the 'naive' concepts of cardinality, and by language the morphological, syntactic and semantic symbols of natural language.

Logic, from the semiotic point of view, is concerned with the types of relations that can arise between the dual qualities of a 'real' world phenomenon. A logical relation as a sign is related to the category of: 1stness, if it signifies all qualities, state and effect, observed and complementary, as independent entities (cf. a, a', b, b'); 2ndness, if it considers the state and effect collections to be independent, but their observed and complementary subsets to be interrelated (cf. A, $\neg A$, B, $\neg B$); 3rdness, if it represents also the state and effect collections as interrelated entities (cf. $A+B$, $A*B$, ..., A is B).

Mathematics *re*-presents the relations found by the logical interpretation of a phenomenon, and introduces common types for the incomparable dual constituents of an interaction. By defining such types, mathematics lays the foundation for the primary concept of language: the relational need [8] (remember that the relational need of a symbol specifies its combinatory potential in a symbol interaction). The mathematical concept of a type defines an induced ordering of (consecutive) numbers. The cognition of such ordering is supported by the brain's potential for *separating* from each other the similar items of the input, and thereby introducing a 'boundary' between such objects (notice that

separation is less meaningful than identification which also contains the meaning of differentiation). The existence of such boundaries is crucial for language, as information about the different items of the input is necessary for determining the reference of a language symbol, for example, a modifier.

Language 'lifts' the three categorical types of logical relations (via the mediation of 'naive' mathematics), respectively, to the concept of a potentially existing, a lexically defined possible, and an actual relational need, characterizing the various types of symbols of natural language.

4.1 Mathematical sign recognition revisited

Finally, let us analyze the representational aspects of mathematical sign recognition, and discuss the relation between mathematical and language signs.

The first 'level' of the recognition of a number-phenomenon consists in the generation of an icon and sinsign representing the observation as some 'thing' and 'event', respectively. As such an icon and sinsign do not include the meaning of the input as a number, this level can be said to be purely *logical*. In language, analogous meanings can arise due to a morphological root (which can be subject to affixation) and an affix (requiring a root), for example.

The second 'level' contains the recognition of the input as an abstract type of number. Such a rheme sign denotes the concept of a general number base (for it does not matter whether houses or dogs are counted), such a legisign a general rule of counting, and such an index sign an increment (type), which is a 'modifier' in virtue of its potential for making the denotation of the number base and the rule of counting more specific, but without changing their primary meaning. This level can be said to be purely *mathematical*, inasmuch its signs signify the observed event only as a mathematical operation, that is, from the *structural* point of view. Language analogues of the two types of index signs are, for example, the adjective and adverb symbols; and, respectively, of the rheme and legisign, are the noun and verb phrases.

The third 'level' includes the recognition of the input as a countable multitude (subject) and a counting algorithm (predicate). This level can be said to be purely *language* related, as the interaction of such subject and predicate may propositionally characterize the meaning of the input as a number-phenomenon. Analogous concepts of natural language are, for example, the syntactic subject and predicate symbols. Let us remember that mathematical subject and predicate signs can only arise if a suitable context (cf. index sign) exists; otherwise, the input is considered to be not well-formed. An example for such a (symbolic) number-phenomenon is "2/1.5" (assuming "/" stands for integer division). The hierarchy of language signs mentioned in this section is displayed in fig. 5.

Summary

In this paper we assume that knowledge arises from the observation of 'real' world phenomena, which are *interactions* between *dual* qualities. This duality is
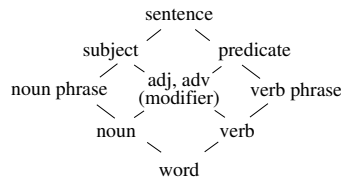
Fig. 5. The hierarchy of language concepts

captured by logic, which represents it only as a *relation*, or a fact. The generalization of the independent qualities of the logical relations as *types* is the main contribution of (naive) mathematics. Language abstracts the mathematical concept of a type, by lifting it to the meaning of a relational need.

We argue that logic, mathematics, and language, are increasingly more meaningful levels of our representational faculty. Mathematical sign recognition includes, in some sense, the whole of this innate ability of the brain. Although it is about mathematical signs only, it shows strong affinity with logical and linguistic signification as well.

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