# Exercises Coalgebra for Lecture 11

The exercises labeled with **(*)** are optional and more advanced.

1. Consider the set $\mathbb{N}^\omega$ of streams over the natural numbers. Let $\mathsf{Rel}_{\mathbb{N}^\omega}$ be the lattice of relations $R \subseteq \mathbb{N}^\omega \times \mathbb{N}^\omega$ on streams over $\mathbb{N}$, ordered by inclusion.

   (a) Define a function $b \colon \mathsf{Rel}_{\mathbb{N}^\omega} \to \mathsf{Rel}_{\mathbb{N}^\omega}$ which captures the stream bisimulations, in the sense that $R \in \mathsf{Rel}_{\mathbb{N}^\omega}$ is a bisimulation iff $R \subseteq b(R)$. Show that your $b$ is monotone (so $R \subseteq S$ implies $b(R) \subseteq b(S)$). What is the greatest fixed point of $b$?

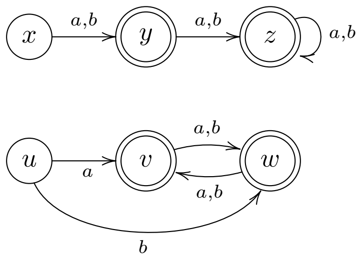
   (b) For $\sigma, \tau \in \mathbb{N}^\omega$, we say $\sigma$ is lexicographically less than $\tau$ if either

      i. $\sigma$ and $\tau$ are equal, or

      ii. $\sigma$ and $\tau$ agree on the first $i$ elements, for some $i$, and $\sigma(i) < \tau(i)$ (we start counting at 0, so this means the $(i+1)$-th element of $\sigma$ is strictly below the $i$-th element of $\tau$).

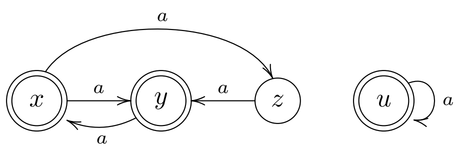
   Define a monotone function $b' \colon \mathsf{Rel}_{\mathbb{N}^\omega} \to \mathsf{Rel}_{\mathbb{N}^\omega}$ such that for any $\sigma, \tau \in \mathbb{N}^\omega$: $\sigma$ is lexicographically less than $\tau$ if and only if $(\sigma, \tau)$ is contained in a relation $R$ such that $R \subseteq b'(R)$.

2. Consider the following deterministic automaton.




   Give a bisimulation up to equivalence that contains $(x, u)$, but which has no more than four pairs in total.

3. Consider the following non-deterministic automaton.




   (a) Draw the determinisation of the automaton.

   (b) The pair $(\{x\}, \{u\})$ is contained in a bisimulation up to congruence (see the notes for the definition; in the lecture, we did an example). What is the smallest one you can find?

4. Let $(P, \leq)$ be a partial order. An element $x \in P$ is called *top* if for all $y \in P$: $y \leq x$. An element $x \in P$ is called *bottom* if for all $y \in P$: $x \leq y$. We typically denote a top element, if it exists, by $\top$, and a bottom element by $\bot$.

   (a) Show that bottom and top elements, if they exist, are unique.

   (b) Show that, if $P$ is a complete lattice, then it has top and bottom elements.

   (c) Prove that $(\mathbb{N}, \leq)$, where $\mathbb{N}$ is the set of natural numbers and $\leq$ is the standard smaller or equal relation on numbers is a poset. Is it a complete lattice?

   (d) Consider now $(\mathbb{N} \cup \{\infty\}, \leq')$, where $\leq'$ is defined as:
      - for all $n \in \mathbb{N} \cup \{\infty\}$: $n \leq' \infty$,
      - for all $n, m \in \mathbb{N}$: $n \leq' m$ iff $n \leq m$.

      Is $(\mathbb{N} \cup \{\infty\}, \leq')$ a poset? Is there a top element $\top$? Is $(N \cup \{\infty\}, \leq')$ a complete lattice?

5. Let $(X, \langle o, \delta)$ be an automaton over an alphabet $A$, and define $b$ and $b'$ as in the lecture:

$$b(R) = \{(x, y) \mid o(x) = o(y) \text{ and for all } a \in A.\ , (\delta(x)(a), \delta(y)(a)) \in R\}$$
$$b'(R) = \{(x, y) \mid o(x) \leq o(y) \text{ and for all } a \in A.\ , (\delta(x)(a), \delta(y)(a)) \in R\}\,.$$

   (a) Show that the equivalence closure $\mathsf{eqv}\colon \mathsf{Rel}_X \to \mathsf{Rel}_X$ is $b$-sound, by showing that it is $b$-compatible: $\mathsf{eqv} \circ b \leq b \circ \mathsf{eqv}$.

   (b) Is $\mathsf{eqv}$ also $b'$-sound? Support your answer with a counterexample or a proof.

6. **(*)** Prove Knaster-Tarski's theorem: if $b\colon P \to P$ is a monotone function on a complete lattice $P$, then $b$ has a greatest fixed point $\mathsf{gfp}(b)$, given by

$$\mathsf{gfp}(b) = \bigvee \{x \in P \mid x \leq b(x)\}\,.$$

   There are some hints in the notes.

7. **(*)** Let $b, f\colon P \to P$ be monotone functions on a complete lattice. A function $f$ is *$b$-compatible* if $f(b(x)) \leq b(f(x))$ for all $x \in X$. In the last part of the notes, it is briefly explained why compatible functions are interesting; here we'll look at a slightly different use of them.

   (a) Show that, if $f$ is $b$-compatible, then $f(\mathsf{gfp}(b)) \leq \mathsf{gfp}(b)$.

   (b) Let $t\colon \mathsf{Rel}_{A^\omega} \to \mathsf{Rel}_{A^\omega}$ be the function defined by $t(R) = R \circ R$, where $R \circ R$ is the relational composition of $R$ with itself. Show that $t$ is $b$-compatible, where $b$ is the function defined in the first part of Exercise 1.

   (c) Similar question to the previous one, but with $b$ replaced by $b'$ from the second part of Exercise 1. What can you conclude about lexicographic order of streams, from (a)?