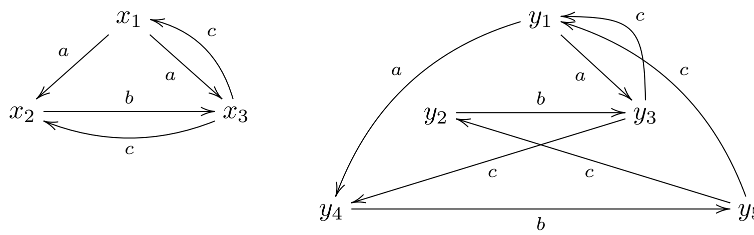
Exercises Coalgebra for Lecture 7

The exercises labeled with **(*)** are optional and more advanced.

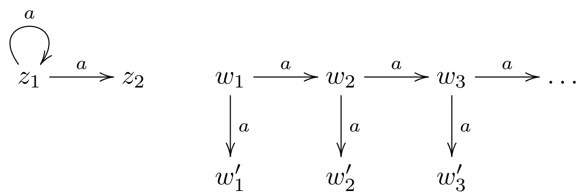
1. For each of the following transition systems and given pairs of states, decide whether they are bisimilar or not (and justify your answer). Are they trace equivalent?

   (a) x_1 and y_1 in:[1]



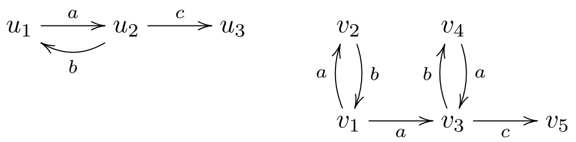
(1)

   (b) z_1 and w_1 in:



(2)

   (c) u_1 and v_1 in:



(3)

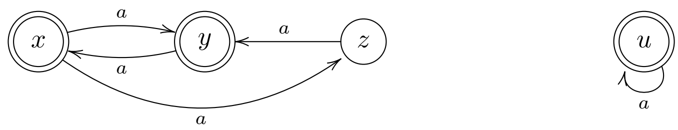
2. Let $\delta \colon X \to (\mathcal{P}(X))^A$ be a labelled transition system. The *trace semantics* $\mathsf{tr} \colon X \to 2^{A^*}$ was defined in the lecture.

   (a) Define a deterministic automaton $\langle o^\sharp, \delta^\sharp \rangle \colon \mathcal{P}(X) \to 2 \times (\mathcal{P}(X))^A$ such that the language accepted by a state $\{x\}$ is given by $l(x)$.

   (b) **(*)** Show that the above construction defines a functor from the category of transition systems (over A) to the category of deterministic automata (over A). What does this say about the relationship between bisimilarity and trace semantics?

[1]Exercise taken from: D. Sangiorgi, Introduction to Bisimulation and Coinduction, Cambridge Universty Press.

3. Prove that x and u in the following non-deterministic automaton accept the same language, using determinisation and bisimulation.



4. Show that if two states of an LTS are bisimilar, then they are also trace equivalent.

5. In the lecture, we defined the determinisation of a non-deterministic automaton $\langle o, \delta \rangle \colon X \to 2 \times (\mathcal{P}_f(X))^A$ as the automaton $\langle o^\sharp, \delta^\sharp \rangle \colon \mathcal{P}_f(X) \to 2 \times (\mathcal{P}_f(X))^A$ given by

$$o^\sharp(S) = \bigvee_{x \in S} o(x) \tag{4}$$

$$\delta^\sharp(S)(a) = \bigcup_{x \in S} \delta(x)(a)\,. \tag{5}$$

Let $\mathsf{beh} \colon \mathcal{P}_f(X) \to 2^{A^*}$ be the language semantics of determinisation. Show that beh is a semilattice homomorphism, i.e., that $\mathsf{beh}(\emptyset)(w) = 0$ and $\mathsf{beh}(S \cup T)(w) = \mathsf{beh}(S)(w) \vee \mathsf{beh}(T)(w)$ for all $S, T \in \mathcal{P}_f(X)$ and $w \in A^*$.

6. (a) Show that semilattices and semilattice homomorphisms form a category SL.
   (b) Does SL have an initial object? And a final object? And products, coproducts?
   (c) Show that there is a 'forgetful' functor $U \colon \mathsf{SL} \to \mathsf{Set}$, mapping a semilattice to the underlying set.
   (d) **(*)** Let $F \colon \mathsf{Set} \to \mathsf{Set}$ be given by $F(X) = 2 \times X^A$. Show that there is a functor $\overline{F} \colon \mathsf{SL} \to \mathsf{SL}$ such that $U \circ \overline{F} = F \circ U$ (what does this mean, concretely)?
   (e) **(*)** Let $G \colon \mathsf{Set} \to \mathsf{Set}$ be given by $G(X) = 2 \times (\mathcal{P}_f(X))^A$. Show that determinisation gives rise to a functor $\mathsf{CoAlg}(G) \to \mathsf{CoAlg}(\overline{F})$.