Exercises Coalgebra for Lecture 13

The exercises labeled with **(*)** are optional and more advanced.

1. Let $(X, \to)$ be a labelled transition system over a set of labels A. Define, by induction, a predicate which holds for every process $p \in X$ such that all paths eventually end up in a stopped state (i.e., with no more transitions). Give your answer both in terms of inference rules, and as the least fixed point of a monotone function on a complete lattice.

2. Consider the lattice $\mathcal{P}(\mathbb{Z}^* \cup \mathbb{Z}^\omega)$ of (finite) lists and (infinite) streams over the integers $\mathbb{Z}$. We write $i : \sigma$ for concatenation of a list or stream σ and an $i \in \mathbb{Z}$. Consider the following inference rules.

$$\frac{\mathsf{pos}(\sigma) \quad i > 0}{\mathsf{pos}(i : \sigma)} \qquad \frac{}{\mathsf{pos}(\langle\rangle)}$$

where $\langle\rangle$ is the empty list.

   (a) Rephrase these inference rules as a monotone function on the lattice $\mathcal{P}(\mathbb{Z}^* \cup \mathbb{Z}^\omega)$.

   (b) What is a pre-fixed point of your function? What is the least pre-fixed point?

   (c) What is a post-fixed point of your function? What is the greatest post-fixed point?

3. In the lecture, we revisited the standard proof principle for induction over natural numbers (prove a property P by showing $P(0)$ and for all n: $P(n) \to P(n+1)$), in terms of pre-fixed points in a lattice. It is customary to weaken the induction step as follows: if $P(i)$ for all $i \leq n$, then $P(n+1)$. Reformulate this in terms of pre-fixed points and the lattice $\mathcal{P}(\mathbb{N})$.

4. Compute $\mathsf{Rel}_B(R)$ for $B \colon \mathsf{Set} \to \mathsf{Set}$ given by

   (a) $B(X) = 2 \times X^A$

   (b) $B(X) = \mathcal{P}(A \times X)$.

   Derive, from the second answer, a notion of bisimulation between transition systems, seen as coalgebras $f \colon X \to \mathcal{P}(A \times X)$.

5. **(*)** Let $G = (V, E)$ be an undirected graph.

   (a) Define a monotone function $b \colon \mathcal{P}(V) \to \mathcal{P}(V)$ such that the greatest fixed point of b consists of all nodes that have an infinite path through the graph (possibly using cycles).

   (b) König's tree lemma states that if $r \in V$ is the root of an infinite tree where each node has finitely many children, then there is an infinite path from r. Prove this lemma by coinduction, using the answer to the first question.

6. **(*)** Complete the proof of the theorem stating that R is a Hermida-Jacobs bisimulation iff it is an Aczel-Mendler bisimulation.

7. **(*)** Consider the category Rel, where an object is a pair (R, X) such that $R \subseteq X \times X$, and an arrow $f\colon (R, X) \to (S, Y)$ is a function $f\colon X \to Y$ such that $(x, y) \in R \Rightarrow (f(x), f(y)) \in S$.

   (a) Let $B\colon \mathsf{Set} \to \mathsf{Set}$ be a functor. Show that relation lifting Rel_B, as we've seen it in the lecture, extends to a functor $\mathsf{Rel}_B\colon \mathsf{Rel} \to \mathsf{Rel}$.

   (b) Show that a Rel_B-coalgebra is the same thing as a bisimulation.