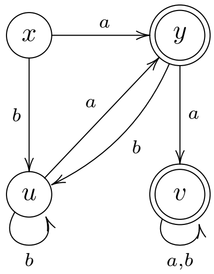
Exercises Coalgebra for Lecture 14

The exercises labeled with **(*)** are optional and more advanced.

1. Use the partition refinement approach from the lecture to compute the greatest bisimulation on the following automaton.

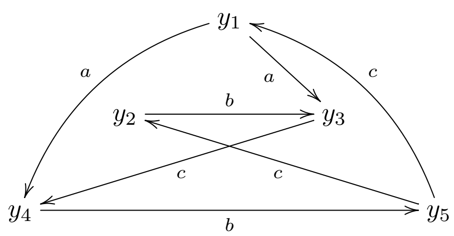


2. Let $(X, \to)$ be a labelled transition system over a set of labels A.

   (a) Define $b\colon \mathsf{Rel}_X \to \mathsf{Rel}_X$ such that R is a bisimulation iff $R \subseteq b(R)$.

   (b) Show that if $R \subseteq X \times X$ is an equivalence relation, then $b(R)$ is an equivalence relation as well.

   (c) Use the final sequence $X \times X \supseteq b(X \times X) \supseteq \ldots$ to compute the greatest bisimulation on the following transition system. Present the relations at each step in terms of partitions.



3. Consider the following monotone function $b\colon \mathsf{Rel}_{\mathbb{N}^\omega} \to \mathsf{Rel}_{\mathbb{N}^\omega}$:

$$b(R) = \{(\sigma, \tau) \mid \sigma(0) \leq \tau(0) \text{ and } (\sigma', \tau') \in R\}\,.$$

   (a) Show that b is cocontinuous.

   (b) Give a concrete description of $b^i(\mathbb{N}^\omega \times \mathbb{N}^\omega)$; prove your claim by induction.

   (c) By the Kleene fixed point theorem, $\mathsf{gfp}(b) = \bigwedge_{i\in\mathbb{N}} b^i(\mathbb{N}^\omega \times \mathbb{N}^\omega)$. Use this to give a concrete description of $\mathsf{gfp}(b)$.

4. Consider the functor $B\colon \mathsf{Set} \to \mathsf{Set}$, $B(X) = A \times X + 1$.

   (a) Draw the first few elements of the final sequence of B.

(b) Give a concrete description of the i-th element $B^i(1)$ of the final sequence.

(c) **(*)** Find the limit of this sequence.

5. **(*)** Show that every cocontinuous function on a complete lattice is also monotone, and show that the converse does not hold.

6. **(*)** Spell out the notions of continuous function; formulate and prove the Kleene fixed point theorem for computing the least fixed point of a continuous function.