Exercises Coalgebra for Lecture 2

1. Exercises 2.5, 2.13, 2.15 from [Rutten05] (see the course website for a link)
2. Consider the function $\mathit{double}\colon A^\omega \to A^\omega$ given by

$$\mathit{double}(\sigma) = (\sigma(0), \sigma(0), \sigma(1), \sigma(1), \ldots)\,.$$

   Define *double* by stream differential equations.

3. Prove, by defining suitable bisimulations, that for all $\sigma, \tau \in A^\omega$:
   (a) $\mathit{odd}(\sigma) = \mathit{even}(\sigma')$
   (b) $\mathit{double}(\sigma) = \mathit{zip}(\sigma, \sigma)$
   (c) $\mathit{odd}(\mathit{zip}(\sigma, \tau)) = \tau$ (Exercise 2.19 in [Rutten05])
4. Complete the proof of Theorem 2.14 (Coinduction; numbering refers to [Rutten05]): show $(\sigma^{(i)}, \tau^{(i)}) \in R$ for all i (see the proof in [Rutten05] for details).
5. In the lecture, we have seen that for every stream system (P, o, t), there exists a map $h\colon P \to A^\omega$, which we claimed to be a homomorphism from (P, o, t) to $(A^\omega, (-)(0), (-)')$ (recall that $(\sigma)(0) = \sigma(0)$ and $(\sigma)' = \sigma'$). This map h is given by

$$h(x) = (o(x), o(t(x)), o(t(t(x))), \ldots)\,.$$

   Formally, it is defined by $h(x)(i) = o(t^i(x))$, where $t^0(x) = x$ and $t^{i+1}(x) = t(t^i(x))$.

   Finish the proof that $(A^\omega, (-)(0), (-)')$ is a final stream system, by showing:

   (a) h, as defined above, is indeed a homomorphism;
   (b) if k, l are homomorphisms from (P, o, t) to $(A^\omega, (-)(0), (-)')$ then $k = l$. Hint: show that $k(x) = l(x)$ for all $x \in P$, by constructing a suitable bisimulation.

6. Consider the stream system (A^ω, o, t) defined by

$$o(\sigma) = \sigma(0) \qquad\qquad t(\sigma) = \sigma''\,.$$

   By the previous exercise, there is a unique homomorphism $h\colon A^\omega \to A^\omega$ from (A^ω, o, t) to $(A^\omega, (-)(0), (-)')$. Spell out the details: what does it mean for h to be a homomorphism? What is h?