Exercises Coalgebra for Lecture 3

1. In the lecture, we have seen the *powerset functor*, which maps a set X to the powerset $\mathcal{P}(X)$, and a function to the direct image.

   (a) Show that the *finite powerset* $\mathcal{P}_f$, defined on a set X by

   $$\mathcal{P}_f(X) = \{U \subseteq X \mid U \text{ is finite}\}$$

   is also a functor (prove all the necessary properties).

   (b) How about the non-empty powerset $\mathcal{P}_{ne}(X) = \{U \subseteq X \mid U \text{ is nonempty}\}$?

   (c) Consider the functor F given on sets as the powerset: $F(X) = \mathcal{P}(X)$ but on a function $f\colon X \to Y$ by

   $$\begin{aligned} F(f)\colon \mathcal{P}(X) &\to \mathcal{P}(Y) \\ U &\mapsto \{y \in Y \mid y \notin \{f(x) \mid x \in X \backslash U\}\} \end{aligned}$$

   Is this a functor? Justify your answer (give a proof or a counterexample).

2. Let $F, G\colon \mathsf{Set} \to \mathsf{Set}$ be functors. Prove that the coproduct $F + G$, given on a set X by $(F+G)(X) = F(X)+G(X)$ is again a functor. Prove that the composition $F \circ G$, given on a set X by $(F \circ G)(X) = F(G(X))$ is a functor.

3. Let $f\colon X \to Y$ be a function. Show that

   (a) f is surjective if and only if there is a function $s\colon Y \to X$ such that $f \circ s = \mathsf{id}$.

   (b) Let $F\colon \mathsf{Set} \to \mathsf{Set}$ be a functor. Show that, if f is surjective, then $F(f)$ is surjective as well.

   (c) How about injective functions? Can you prove something similar?

4. Let A, B, C be sets, and consider the (exponent) polynomial functor given on sets by $F(X) = A+(B\times X)^C$. How is F defined on functions, according to the definition of polynomial functions given in the lecture/section 2.2 of the book?

5. In the lecture, we have briefly seen that non-deterministic automata over an alphabet A are coalgebras for the functor F given by $F(X) = \{0,1\} \times (\mathcal{P}(X))^A$. Explain why. What happens (in terms of automata) if we replace $\mathcal{P}$ by $\mathcal{P}_f$? And by $\mathcal{P}_{ne}$ (as defined in the first exercise)?

6. We have seen that stream systems over A are coalgebras for the functor F given by $F(X) = A \times X$.

   (a) What are coalgebras for the functor G given by $G(X) = (A \times X) + 1$ (defined on a function f as $G(f) = (\mathsf{id}_A \times f) + \mathsf{id}_1$), where $1 = \{*\}$ is a one-element set?

(b) What is a homomorphism between such G-coalgebras? (Instantiate the general definition of coalgebra homomorphisms to the functor G, and spell out the details)

(c) How about coalgebras for the functor H given by $H(X) = (A \times X) + B$?

7. A *labelled transition system* over a set of labels A consists of a set X and a relation $\rightarrow \subseteq X \times A \times X$. We write $x \xrightarrow{a} y$ if $(x, a, y) \in \rightarrow$.

   (a) Show that labelled transition systems are coalgebras for the functor F given by $F(X) = \mathcal{P}(A \times X)$.

   (b) What is a homomorphism from one labelled transition system to another? Instantiate the definition of coalgebra homomorphism for the above functor F, and spell out the details (in terms of transitions $x \xrightarrow{a} y$).

   (c) A labelled transition system $(X, \rightarrow)$ is *finitely branching* if, for each $x \in X$, the set $\{(a, y) \mid (x, a, y) \in \rightarrow\}$ is finite. Can you capture finitely branching labelled transition systems as coalgebras?

   (d) A labelled transition system $(X, \rightarrow)$ is *image-finite* if, for each $x \in X$ and $a \in A$, the set $\{y \mid (x, a, y) \in \rightarrow\}$ is finite. Can you capture image-finite labelled transition systems as coalgebras?