Exercises Coalgebra for Lecture 4

The exercises labeled with (*) are optional and more advanced.

1. We would like to define a category Pred, as follows: an object of Pred is a pair (P, X) of sets such that $P \subseteq X$, and an arrow from an object (P, X) to an object (Q, Y) is a function $f\colon X \to Y$ such that for all $x \in P$: $f(x) \in Q$. Show that Pred is a category, by defining suitable identity arrows and composition of arrows, and checking that the required laws are satisfied.

2. Let $\mathcal{C}$ be a category. We define the *opposite category* $\mathcal{C}^{\mathsf{op}}$ as the category which has the same objects as $\mathcal{C}$, but where all arrows are reversed: thus, $f\colon X \to Y$ is an arrow in $\mathcal{C}^{\mathsf{op}}$ iff $f\colon Y \to X$ is an arrow in $\mathcal{C}$.

   (a) How should composition in $\mathcal{C}^{\mathsf{op}}$ be defined? And identity arrows? Show, in detail, that $\mathcal{C}^{\mathsf{op}}$ is a category.

   (b) Show that an object 0 is initial in $\mathcal{C}$ iff it is final in $\mathcal{C}^{\mathsf{op}}$.

3. (*) A *monoid* is a triple $(M, \cdot, 1)$ where M is a set, $\cdot$ is a binary operation and $1 \in M$ an element, such that for all $m, n, p \in M$: $(m \cdot n) \cdot p = m \cdot (n \cdot p)$ and $m \cdot 1 = m = 1 \cdot m$.

   (a) Show that a monoid corresponds to a one-object category.

   (b) Let $(M, \cdot, 1)$ be a monoid, represented as a category M as in the previous exercise. Show that a functor $F\colon M \to \mathsf{Set}$ corresponds to a *monoid action*: a set X together with a function $\mu\colon M \to X^X$ (where X^X is the set of functions from X to X) such that for all $x \in X$: $\mu(1)(x) = x$ and for all $m, n \in M$: $\mu(m \cdot n)(x) = \mu(m)(\mu(n)(x))$.

4. What are initial/final objects in the following categories (if they exist)?

   (a) $\mathsf{SetsRel}$ (recall: objects are sets, arrows are relations);

   (b) the category $\mathbb{N}$, whose objects are natural numbers, and where there is a (single) arrow from n to m iff $n \leq m$;

   (c) the *discrete category* for a given set X; objects are elements of X, and the only arrows are the identity arrows;

   (d) (*) the category Mon whose objects are monoids (see 3) and whose arrows are monoid homomorphisms; a homomorphism from $(M, \cdot_M, 1_M)$ to $(N, \cdot_N, 1_N)$ is a function $h\colon M \to N$ such that for all $m, n \in M$: $h(m \cdot_M n) = h(m) \cdot_N h(n)$ and $h(1_M) = 1_N$.

5. Recall that two objects X, Y in a category $\mathcal{C}$ are *isomorphic* if there exist arrows $f\colon X \to Y$ and $g\colon Y \to X$ such that $g \circ f = \mathsf{id}_X$ and $f \circ g = \mathsf{id}_Y$. Show that any functor $F\colon \mathcal{C} \to \mathcal{D}$ preserves isomorphisms: if X, Y are isomorphic then $F(X)$ and $F(Y)$ are isomorphic as well.

6. Let $F\colon \mathcal{C} \to \mathcal{C}$ be an (endo)functor on an arbitrary category $\mathcal{C}$. In the lecture, we defined the category $\mathsf{CoAlg}(F)$, whose objects are F-coalgebras and whose arrows are F-coalgebra homomorphisms. Show that this is indeed a category, by showing, with a detailed (equational) calculation, that

   (a) the identity arrow $\mathsf{id}_X\colon X \to X$ on the carrier of any F-coalgebra (X, f) is a homomorphism, and

   (b) if h is a homorphism from a coalgebra (X_1, f_1) to a coalgebra (X_2, f_2), and k is a homomorphism from a coalgebra (X_2, f_2) to a coalgebra (X_3, f_3), then $k \circ h$ is a homomorphism from (X_1, f_1) to (X_3, f_3). Hint: organize these homomorphisms into a nice diagram.

7. **(*)** Let $F\colon \mathcal{C} \to \mathcal{C}$ be a functor on a category $\mathcal{C}$. Show that, if $\mathcal{C}$ has an initial object 0, then $\mathsf{CoAlg}(F)$ has an initial object as well.