Exercises Coalgebra for Lecture 5

The exercises labeled with **(*)** are optional and more advanced.

1. Let A be a set, and consider the functor $F\colon \mathsf{Set} \to \mathsf{Set}$, defined on a set X by $F(X) = A \times X \times X$ and on a function f by $F(f) = \mathsf{id}_A \times f \times f$. An *infinite binary tree* (node-labelled in A) is a function $t\colon \{l, r\}^* \to A$, where $\{l, r\}^*$ is the set of words over l, r. The empty word is denoted by $\varepsilon \in \{l, r\}^*$. The set $A^{\{l,r\}^*}$ of all infinite binary trees is denoted by T.

   Given a tree $t \in T$, we define $t_l, t_r \in T$ as follows: $t_l(w) = t(lw)$ and $t_r(w) = t(rw)$, for all $w \in \{l, r\}^*$. Consider the F-coalgebra $z\colon T \to A \times T \times T$ defined by
   $$z(t) = (t(\varepsilon), t_l, t_r)$$
   The aim of this exercise is to show that (T, z) is a final F-coalgebra.

   (a) Describe, in words, what t_l and t_r are, given a tree $t \in T$.

   (b) Let $g\colon X \to A \times X \times X$ be an F-coalgebra. Show that a map $h\colon X \to T$ is a homomorphism from (X, g) to (T, z) if and only if for all $x \in X$: if $g(x) = (a, y, z)$, then

   i. $h(x)(\varepsilon) = a$

   ii. $h(x)_l = h(y)$

   iii. $h(x)_r = h(z)$

   (c) The aim is to define a homomorphism from a given F-coalgebra (X, g) to (T, z). Given $x \in X$ and $w \in A^*$, we let $g(x) = (a, y, z)$ and we would like to define:
   $$h(x)(\varepsilon) = \ldots$$
   $$h(x)(lw) = \ldots$$
   $$h(x)(rw) = \ldots$$
   Fill in the dots, such that h is a homomorphism.

   (d) Show that if h, k are both homomorphisms from (X, g) to (T, z) then $h = k$, by proving by induction on $w \in \{l, r\}^*$ that for all $x \in X$: $h(x)(w) = k(x)(w)$.

2. Recall that $z\colon A^\omega \to A \times A^\omega$ defined by $z(\sigma) = (\sigma(0), \sigma')$, is the final F-coalgebra, where $F(X) = A \times X$.

   Define $\mathsf{even}\colon A^\omega \to A^\omega$ (which returns the stream consisting of elements at the even positions of the input) as the unique homomorphism from some suitable F-coalgebra to the final F-coalgebra. Similarly for odd, and for $\mathsf{tail}\colon A^\omega \to A^\omega$, where $\mathsf{tail}(\sigma)(n) = \sigma(n + 1)$.

3. **(*)** Let $F\colon \mathsf{Set} \to \mathsf{Set}$ be a functor with a final coalgebra (Z, z). In the lecture, we defined two states $x, y \in X$ of an F-coalgebra (X, f) to

be *behaviourally equivalent* if $\mathsf{beh}(x) = \mathsf{beh}(y)$, where beh is the unique homomorphism from (X, f) to (Z, z). Show that $\mathsf{beh}(x) = \mathsf{beh}(y)$ if and only if there exists an F-coalgebra (Y, g) and a homomorphism $h \colon X \to Y$ from (X, f) to (Y, g) such that $h(x) = h(y)$. Hint: draw a suitable diagram to clarify the situation.

4. **(*)** Consider the functor $F \colon \mathsf{Set} \to \mathsf{Set}$, defined on a set X by $F(X) = X + 1$, where $1 = \{*\}$ is a singleton. Give a final coalgebra for F.

5. **(*)** We would like to define a functor $S \colon \mathsf{Set} \to \mathsf{Set}$ by $S(X) = X^\omega$, i.e., a set X is mapped to the set of streams over X.

   (a) Define S on a function $f \colon X \to Y$, using that Y^ω is the final stream system over Y; $S(f)$ should apply f to all elements of a given stream.

   (b) Show that S is functorial.